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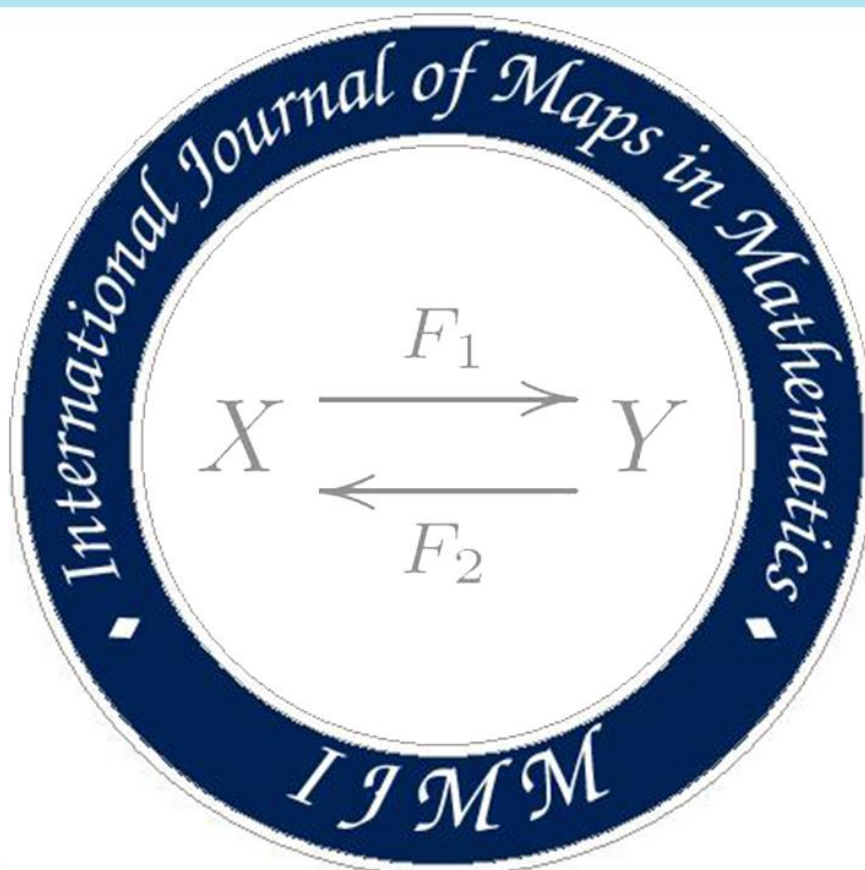
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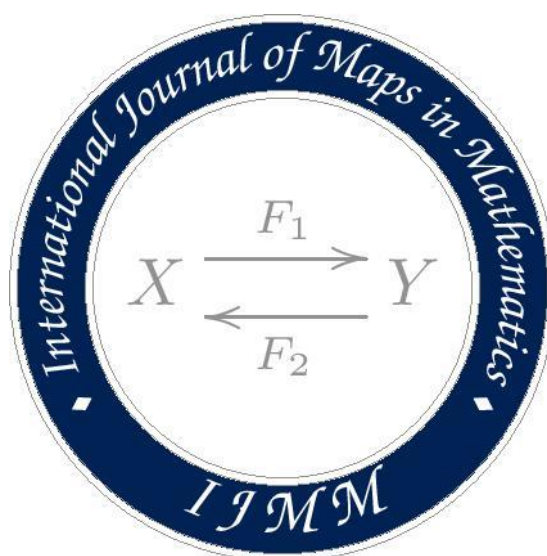
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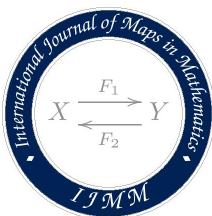
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ON A GENERALIZED SUBCLASS OF MEROMORPHIC p -VALENT CLOSE TO CONVEX FUNCTIONS IN q -ANALOGUE.

BAKHTIAR AHMAD* AND MUHAMMAD ARIF

ABSTRACT. In this article, we define a new subclass of meromorphic multivalent close to convex functions involving in q -calculus associated with Janowski functions. We investigate some useful geometric properties such as sufficiency criteria, distortion problem, growth theorem, radii of starlikeness and convexity and coefficient estimates for this class.

1. INTRODUCTION

The q -calculus has motivated the researchers in the recent past due to its numerous physical and mathematical applications. The generalization of derivative and integral in q -calculus which are known as q -analogue of derivative and integral were introduced and studied by Jackson [11, 12]. Aral and Gupta [5, 6] used some what similar concept and defined q -Baskakov Durrmeyer operator by using q -beta function. Similarly the author's in [3, 7] generalized some complex operators, which are known as q -Picard and q -Gauss-Weierstrass singular integral operators. Later, Srivastava and Bansal [20, pp. 62] used the q -analogue of derivative in Geometric function theory by introducing the q -generalization of starlike functions for the first time, see also [19, pp. 347 *et seq.*].

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In 2014, the q -analogue of Ruscheweyh operator were studied by Kanas and Răducanu [14], and they investigated some of its properties as well. The applications of this differential operator were further studied by Mohammed and Darus [2] and Mahmood and Sokół [15]. In this article we introduce a subclass of meromorphic multivalent functions in association with Janowski functions and study its geometric properties like sufficiency criteria, inclusion property, coefficient bounds, radii problem and distortion theorem.

2. PRELIMINARIES AND DEFINITIONS

Let \mathfrak{A}_p denote the class of all meromorphic multivalent functions $f(z)$ that are analytic in the punctured disc $\mathbb{D} = \{z \in \mathbb{C} : 0 < |z| < 1\}$ and satisfying the normalization

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (z \in \mathbb{D}). \quad (2.1)$$

The q -derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \quad (z \neq 0), \quad (2.2)$$

where $0 < q < 1$. It can easily be seen that for $n \in \mathbb{N}$ and $z \in \mathbb{D}$

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q] a_n z^{n-1}, \quad (2.3)$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^n q^l, \quad [0, q] = 0.$$

For any non-negative integer n the q -number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, & n = 0, \\ [1, q] [2, q] [3, q] \cdots [n, q], & n \in \mathbb{N}. \end{cases}$$

The Subordination concept has been utilized in the introduction of our new class which can be defined as

Definition 2.1. If $h_1(z)$ and $h_2(z)$ are two functions both analytic in E , then $h_1(z) \prec h_2(z)$, and we say that $h_1(z)$ is subordinated to $h_2(z)$, while there is an analytic function $w(z)$ which is known as Schwarz function and satisfy the conditions $|w(z)| < 1$ and $w(0) = 0$ ($z \in E$), imply that $h_1(z) = h_2(w(z))$. Especially, for a univalent function $h_2(z)$ this subordination is equivalent to $h_1(E) \subseteq h_2(E)$ and $h_1(0) = h_2(0)$.

Motivated from the work discussed above and studied in [8, 10, 13, 17, 18, 21, 23], we now define a new subclass $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ of \mathfrak{A} as follows;

Definition 2.2. Let $-1 \leq B < A \leq 1$ and $0 < q < 1$. Then a function $f \in \mathfrak{A}$ is in the class $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if it satisfies

$$-\frac{z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z)g(tz)} \prec \frac{p + [pB + (p - \alpha)(A - B)]z}{p(1 + Bz)}. \quad (2.4)$$

where $g(z)$ is in the class $\mathcal{MS}_p^*(1/2)$.

$$F_\delta(z) = \frac{(1 - \delta)[p, q]f(z) - \delta z \partial_q f(z)}{[p, q]}$$

and the notation " \prec " denotes the familiar subordinations.

We note that

- (1) For $A = 1, B = -1, \delta = 0$ and $q \rightarrow 1^-$ we get $\mathcal{MK}_p(\alpha)$ the class of meromorphic multivalent close to convex functions order α .
- (2) For $A = 1, B = -1, \delta = 0, \alpha = 0$ and $q \rightarrow 1^-$ we get \mathcal{MK}_p the class of meromorphic multivalent close to convex functions.
- (3) For $A = 1, B = -1, \delta = 0, p = 1$ and $q \rightarrow 1^-$ we get \mathcal{MK} the class of meromorphic close to convex functions of order α .

Equivalently a function $f(z) \in \mathfrak{A}$ is in the class $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if and only if

$$\left| \frac{\frac{z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z)g(tz)} + 1}{B + (1 - \frac{\alpha}{p})(A - B) + B \frac{z^{1-p}\partial_q F_\delta(z)}{[p,q]t^p g(z)g(tz)}} \right| < 1. \quad (2.5)$$

For our main results we will need the following.

Lemma 2.1. [22] Let

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n \prec k(z) = 1 + \sum_{n=1}^{\infty} k_n z^n$$

in \mathbb{D} . If $k(z)$ is univalent in \mathbb{D} and $k(\mathbb{D})$ is convex, then

$$|d_n| \leq |k_1|, \text{ for } n \geq 1.$$

Theorem 2.1. [4] Let $g_i(z) \in \mathcal{MS}_p^*(\alpha_i)$ with $i = 1, 2$. Then

$$t_1^p t_2^p z^p g_1(t_1 z) g_2(t_2 z) \in \mathcal{MS}_p^*(\gamma),$$

where $\gamma = \alpha_1 + \alpha_2 - 1$ and $0 < |t_i| \leq 1$.

Now for $t_1 = 1, t_2 = t$ and $g_1(z) = g_2(z) = g(z)$ we get

Corollary 2.1. If $g(z) \in \mathcal{MS}_p^*(1/2)$ then $G(z) = t^p z^p g(z) g(tz) \in \mathcal{MS}_p^*(0) = \mathcal{MS}_p^*$.

3. MAIN RESULTS

In this Section we start with sufficiency criteria for this class in the following theorem.

Theorem 3.1. *Let $f \in \mathfrak{A}$ be of the form (2.1). Then the function $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if and only if the following inequality holds*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) (1+B) [n+p, q] |a_{n+p}| + (1+B) \right. \\ \left. + (1 - \frac{\alpha}{p})(A-B) \frac{2p[p,q]}{p+n} \right) \leq (1 - \frac{\alpha}{p})(A-B) [p, q]. \end{aligned} \quad (3.6)$$

Proof. Let us suppose that the first inequality (3.6) holds. Then to show that $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, we only need to prove the inequality (2.5). For this consider

$$\left| \frac{\frac{z \partial_q F_{\delta}(z)}{[p,q]G(z)} + 1}{B + (1 - \frac{\alpha}{p})(A-B) + B \frac{z \partial_q F_{\delta}(z)}{[p,q]G(z)}} \right| = \left| \frac{z \partial_q F_{\delta}(z) + [p,q]G(z)}{(B + (1 - \frac{\alpha}{p})(A-B)) [p,q]G(z) + B z \partial_q F_{\delta}(z)} \right|.$$

Now with the help of (2.2), (2.3), (2.1) and

$$G(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}, \quad (z \in \mathbb{D}), \quad (3.7)$$

we have

$$\begin{aligned} &= \left| \frac{-\frac{[p,q]}{z^p} + \sum_{n=1}^{\infty} \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] a_{n+p} z^{n+p} + \frac{[p,q]}{z^p} + [p,q] \sum_{n=1}^{\infty} b_{n+p} z^{n+p}}{(B + (1 - \frac{\alpha}{p})(A-B)) \left(\frac{[p,q]}{z^p} + [p,q] \sum_{n=1}^{\infty} b_{n+p} z^{n+p} \right) + B \left(-\frac{[p,q]}{z^p} + \sum_{n=1}^{\infty} \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] a_{n+p} z^{n+p} \right)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] a_{n+p} + [p,q] b_{n+p} \right) z^{n+p}}{\frac{(1 - \frac{\alpha}{p})(A-B)[p,q]}{z^p} + \sum_{n=1}^{\infty} \left(B \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] a_{n+p} + (B + (1 - \frac{\alpha}{p})(A-B)) [p,q] b_{n+p} \right) z^{n+p}} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] a_{n+p} + [p,q] b_{n+p} \right) z^{n+2p}}{(1 - \frac{\alpha}{p})(A-B)[p,q] + \sum_{n=1}^{\infty} \left(B \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] a_{n+p} + (B + (1 - \frac{\alpha}{p})(A-B)) [p,q] b_{n+p} \right) z^{n+2p}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] |a_{n+p}| + [p,q] |b_{n+p}| \right)}{(1 - \frac{\alpha}{p})(A-B)[p,q] - \sum_{n=1}^{\infty} \left(B \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] |a_{n+p}| + (B + (1 - \frac{\alpha}{p})(A-B)) [p,q] |b_{n+p}| \right)} \end{aligned}$$

As $g(z) \in \mathcal{MS}_p^*(1/2)$ then by corollary 2.1 $G(z)$ is in the class \mathcal{MS}_p^* with representation (3.7) then by [24]

$$|b_{p+n}| \leq \frac{2p}{p+n} \quad (3.8)$$

we get

$$\begin{aligned} &\leq \frac{\sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] |a_{n+p}| + \frac{2p[p,q]}{p+n} \right)}{(1 - \frac{\alpha}{p})(A-B)[p,q] - \sum_{n=1}^{\infty} \left(B \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) [n+p, q] |a_{n+p}| + (B + (1 - \frac{\alpha}{p})(A-B)) \frac{2p[p,q]}{p+n} \right)} \\ &< 1 \end{aligned}$$

where we have used the inequality (3.6) and this completes the direct part.

Conversely, let $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ be given by (2.1). Then from (2.5), we have for $z \in \mathbb{D}$,

$$\begin{aligned} & \left| \frac{\frac{z \partial_q F_\delta(z)}{[p,q]G(z)} + 1}{B + (1 - \frac{\alpha}{p})(A - B) + B \frac{z \partial_q F_\delta(z)}{[p,q]G(z)}} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q] - \delta[p+n,q]}{[p,q]} \right) [n+p,q] a_{n+p} + [p,q] b_{n+p} \right) z^{n+2p}}{(1 - \frac{\alpha}{p})(A - B)[p,q] + \sum_{n=1}^{\infty} \left(B \left(\frac{(1-\delta)[p,q] - \delta[p+n,q]}{[p,q]} \right) [n+p,q] a_{n+p} + \left(B + (1 - \frac{\alpha}{p})(A - B) \right) [p,q] b_{n+p} \right) z^{n+2p}} \right| \end{aligned}$$

Since $|\Re z| \leq |z|$, we have

$$\begin{aligned} \Re \left\{ \frac{\sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q] - \delta[p+n,q]}{[p,q]} \right) [n+p,q] a_{n+p} + [p,q] b_{n+p} \right) z^{n+2p}}{(1 - \frac{\alpha}{p})(A - B)[p,q] + \sum_{n=1}^{\infty} \left(B \left(\frac{(1-\delta)[p,q] - \delta[p+n,q]}{[p,q]} \right) [n+p,q] a_{n+p} + \left(B + (1 - \frac{\alpha}{p})(A - B) \right) [p,q] b_{n+p} \right) z^{n+2p}} \right\} \\ < 1 \end{aligned} \quad (3.9)$$

Now choose values of z on the real axis so that $\frac{z \partial_q F_\delta(z)}{[p,q]G(z)}$ is real. Upon clearing the denominator in (3.9) and letting $z \rightarrow 1^-$ through real values, we obtain (3.6).

Taking $q \rightarrow 1^-$ we get the result.

Corollary 3.1. [4] *Let $f \in \mathfrak{A}$ be of the form (2.1). Then the function $f \in \lim_{q \rightarrow 1^-} \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if and only if the following inequality holds*

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)p - \delta(p+n)}{p} \right) (1 + B)(p + n) |a_{n+p}| + (1 + B \right. \\ \left. + (1 - \frac{\alpha}{p})(A - B) \frac{2p^2}{p+n} \right) \leq (p - \alpha)(A - B). \end{aligned}$$

Now we calculate the coefficients estimates for this newly defined class.

Theorem 3.2. *Let $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ and be of the form (2.1). Then*

$$|a_{p+n}| \leq \frac{[p,q]^2}{[p+n,q]((1-\delta)[p,q] - \delta[p+n,q])} \left(\frac{2p}{p+n} + 2(p - \alpha)(A - B) \sum_{i=2}^{n-1} \frac{1}{p+i} \right).$$

Proof. For $f \in \mathfrak{A}$ is in the class $\mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, if it satisfies

$$\frac{-z^{1-p} \partial_q F_\delta(z)}{[p,q] t^p g(z) g(tz)} \prec \frac{1 + [B + (1 - \frac{\alpha}{p})(A - B)]z}{1 + Bz}.$$

Now if

$$G(z) = t^p z^p g(z) g(tz)$$

and

$$h(z) = \frac{-z \partial_q F_\delta(z)}{[p,q]G(z)}, \quad (3.10)$$

and it will be of the form

$$h(z) = 1 + \sum_{n=1}^{\infty} d_n z^n.$$

Since

$$h(z) \prec \frac{1+[B+(1-\frac{\alpha}{p})(A-B)]z}{1+Bz} = 1 + \frac{(p-\alpha)(A-B)}{p}z + \dots$$

Then by Lemma 2.1 we get

$$|d_n| \leq \frac{(p-\alpha)(A-B)}{p} \quad (3.11)$$

Now putting the series expansions of $h(z)$, $G(z)$ and $f(z)$ in (3.10), simplifying and comparing the coefficients of z^{p+n} on both sides

$$\begin{aligned} -\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]^2} [p+n, q] a_{p+n} &= b_{p+n} + b_{p+n-1}d_1 + \\ &+ b_{p+n-2}d_2 + \dots + b_{p+1}d_{n-1}. \end{aligned}$$

Taking absolute on both sides, using the triangle inequility and then using (3.11) and (3.8) we obtain

$$\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]^2} [p+n, q] |a_{p+n}| \leq \frac{2p}{n+p} + \frac{(p-\alpha)(A-B)}{p} \sum_{i=2}^{n-1} \frac{2p}{p+i},$$

which implies that

$$|a_{p+n}| \leq \frac{[p,q]^2}{[p+n,q]((1-\delta)[p,q]-\delta[p+n,q])} \left(\frac{2p}{p+n} + 2(p-\alpha)(A-B) \sum_{i=2}^{n-1} \frac{1}{p+i} \right).$$

where $|a_1| = 1$ and we get the desired proof.

Taking $q \rightarrow 1^-$ we get the coefficient estimates for the class which was studied by Arif et. al. [4].

Corollary 3.2. *Let $f \in \mathfrak{A}$ be of the form (2.1), and $f \in \lim_{q \rightarrow 1^-} \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$, then*

$$|a_{p+n}| \leq \frac{p^2}{(p+n)((1-\delta)p-\delta(p+n))} \left(\frac{2p}{p+n} + 2(p-\alpha)(A-B) \sum_{i=2}^{n-1} \frac{1}{p+i} \right).$$

The next result is about the distortion theorem for this class of functions.

Theorem 3.3. *If $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$ and has the form (2.1). Then for $|z| = r$*

$$\frac{[p,q](1-Cr)(1-r)^{p+1}}{r^{p+1}(1-Br)} \leq |\partial_q F_\delta(z)| \leq \frac{[p,q](1+Cr)(1+r)^{p+1}}{r^{p+1}(1+Br)}$$

where $C = B + (1 - \frac{\alpha}{p})(A - B)$.

Proof. Suppose that $f \in \mathcal{MK}_{p,q}(\alpha, \delta, A, B)$. Then we can write

$$\frac{-z^1 \partial_q F_\delta(z)}{[p, q]G(z)} \prec \frac{1 + Cz}{1 + Bz}$$

then with $|z| = r$ and

$$\left| \frac{-z^1 \partial_q F_\delta(z)}{[p, q]G(z)} - \frac{1 - CBr^2}{1 - B^2r^2} \right| \leq \frac{(C - B)r}{1 - B^2r^2}.$$

simplification gives us

$$\frac{1 - Cr}{1 - Br} \leq \left| \frac{-z \partial_q F_\delta(z)}{[p, q]G(z)} \right| \leq \frac{1 + Cr}{1 + Br}. \quad (3.12)$$

Now since $G(z) \in \mathcal{MS}_p^*$, thus we have

$$\frac{(1 - r)^{p+1}}{r^p} \leq |G(z)| \leq \frac{(1 + r)^{p+1}}{r^p}. \quad (3.13)$$

Now by using (3.13) in (3.12), we obtain the required result.

In the following we give the growth theorem for this class.

Theorem 3.4. Let $f \in \mathcal{MK}_q^*(p, \mu, A, B)$ and has the form (2.1). Then for $|z| = r$

$$\frac{1}{r^p} - \tau_1 r^p \leq |f(z)| \leq \frac{1}{r^p} + \tau_1 r^p,$$

where

$$\tau_1 = \frac{[p, q]^2 ((p - \alpha)(A - B) - (p(1 + B) + (p - \alpha)(A - B)))}{(p + 1)(1 + B)[p + 1, q]((1 - \delta)[p, q] - \delta[p + 1, q])}.$$

Proof. Consider

$$\begin{aligned} |f(z)| &= \left| \frac{1}{z^p} + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \right|, \\ &\leq \frac{1}{|z^p|} + \sum_{n=1}^{\infty} |a_{n+p}| |z|^{n+p} \\ &= \frac{1}{r^p} + \sum_{n=1}^{\infty} |a_{n+p}| r^{n+p} \end{aligned}$$

As $|z| = r < 1$ so $r^{n+p} < r^p$ and

$$|f(z)| \leq \frac{1}{r^p} + r^p \sum_{n=1}^{\infty} |a_{n+p}| \quad (3.14)$$

Similarly

$$|f(z)| \geq \frac{1}{r^p} - r^p \sum_{n=1}^{\infty} |a_{n+p}| \quad (3.15)$$

Since (3.6) implies that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) (1+B) [n+p, q] |a_{n+p}| + \right. \\ \left. \left(1+B + \left(1 - \frac{\alpha}{p} \right) (A-B) \right) \frac{2p[p,q]}{p+n} \right) \leq \left(1 - \frac{\alpha}{p} \right) (A-B) [p, q]. \end{aligned}$$

But

$$\begin{aligned} (p(1+B) + (p-\alpha)(A-B)) \frac{2[p,q]}{p+1} + \frac{((1-\delta)[p,q]-\delta[p+n,q])[p+1,q](1+B)}{[p,q]} \sum_{n=1}^{\infty} |a_{n+p}| \\ \leq \sum_{n=1}^{\infty} \left(\left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) (1+B) [n+p, q] |a_{n+p}| + \right. \\ \left. \left(1+B + \left(1 - \frac{\alpha}{p} \right) (A-B) \right) \frac{2p[p,q]}{p+n} \right). \end{aligned}$$

Hence

$$\begin{aligned} (p(1+B) + (p-\alpha)(A-B)) \frac{2[p,q]}{p+1} + \frac{((1-\delta)[p,q]-\delta[p+1,q])[p+1,q](1+B)}{[p,q]} \sum_{n=1}^{\infty} |a_{n+p}| \\ \leq \left(1 - \frac{\alpha}{p} \right) (A-B) [p, q], \end{aligned}$$

which gives

$$\sum_{n=1}^{\infty} |a_{n+p}| \leq \frac{[p,q]^2((p-\alpha)(A-B)-(p(1+B)+(p-\alpha)(A-B)))}{(p+1)(1+B)[p+1,q]((1-\delta)[p,q]-\delta[p+1,q])}$$

Now by putting this value in (3.14) and (3.15) we get the required result.

In the next two results we determine the radii of convexity and starlikeness of order σ .

Theorem 3.5. *Let $f \in \mathcal{MK}_q^*(p, \mu, A, B)$. Then $f \in \mathcal{MC}_p(\sigma)$ for $|z| < r_1$, where*

$$r_1 = \left(\frac{p^2(p-\sigma)(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(p+n)(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} \right)^{\frac{1}{n+2p}}.$$

Proof. Let $f \in \mathcal{MK}_q^*(p, \mu, A, B)$. To prove $f \in \mathcal{MC}_p(\sigma)$, we only need to show

$$\left| \frac{zf''(z) + (p+1)f'(z)}{zf''(z) + (1+2\sigma-p)f'(z)} \right| < 1.$$

Using (2.1) along with some simple computation yields

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p+\sigma)}{p(p-\sigma)} |a_{n+p}| |z|^{n+2p} < 1. \quad (3.16)$$

From (3.6), we can easily obtain that

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{(1-\delta)[p,q]-\delta[p+n,q]}{[p,q]} \right) (1+B) [n+p, q] |a_{n+p}| \\ \leq \frac{[p,q]((p-\alpha)(A-B)-2(p(1+B)+(p-\alpha)(A-B))p)}{p(p+1)}. \\ \Rightarrow \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}| < 1. \end{aligned}$$

Now inequality (3.16) will be true, if the following holds

$$\sum_{n=1}^{\infty} \frac{(p+n)(n+p+\sigma)}{p(p-\sigma)} |a_{n+p}| |z|^{n+2p} < \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}|,$$

which implies that

$$|z|^{n+2p} < \frac{p^2(p-\sigma)(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(p+n)(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))},$$

and so

$$|z| < \left(\frac{p^2(p-\sigma)(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(p+n)(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} \right)^{\frac{1}{n+2p}},$$

$$= r_1.$$

we get the required condition.

Theorem 3.6. *Let $f \in \mathcal{MK}_q^*(p, \mu, A, B)$. Then $f \in \mathcal{MS}_p^*(\sigma)$ for $|z| < r_2$, where*

$$r_2 = \left(\frac{(p-\sigma)p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} \right)^{\frac{1}{n+2p}},$$

Proof. We know that $f \in \mathcal{MS}_p^*(\sigma)$, if and only if

$$\left| \frac{zf'(z) + pf(z)}{zf'(z) - (p-2\sigma)f(z)} \right| \leq 1.$$

Using (2.1) and upon simplification yields

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\sigma}{p-\sigma} \right) |a_{n+p}| |z|^{n+2p} < 1. \quad (3.17)$$

Now from (3.6) we can easily obtain

$$\Rightarrow \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}| < 1.$$

For inequality (3.17) to be true it will be enough if

$$\sum_{n=1}^{\infty} \left(\frac{n+p+\sigma}{p-\sigma} \right) |a_{n+p}| |z|^{n+2p} < \sum_{n=1}^{\infty} \frac{p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} |a_{n+p}|.$$

This gives

$$|z|^{n+2p} < \frac{(p-\sigma)p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))},$$

and hence

$$|z| < \left(\frac{(p-\sigma)p(p+1)(1+B)[n+p,q]((1-\delta)[p,q]-\delta[p+n,q])}{(n+p+\sigma)[p,q]^2((p-\alpha)(A-B)-2p((1+B)+(p-\alpha)(A-B)))} \right)^{\frac{1}{n+2p}} = r_2,$$

Thus we obtain the required result.

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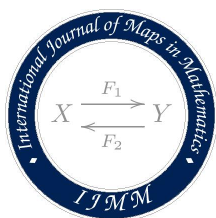
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AN OPTIMUM PARAMETER METHOD TO OBTAIN NUMERICAL SOLUTIONS OF THE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT. The main purpose of this article is to use a method with a free parameter which is named optimum asymptotic homotopy method (*OHAM*) in order to obtain the solution of differential equations, partial differential equations and the system of coupled partial differential equations featuring fractional derivative. This method is preferable to others since it has faster convergence toward homotopy perturbation method as well as the convergence rate can be set as controlled area. Various examples are given to better understand the use of this method. The approximate solutions are compared with exact solutions as well.

1. INTRODUCTION

Fractional arithmetic and fractional differential equations appeared in many disciplines, including medicine [1], economics [2], dynamical problems [3, 4], chemistry [5], mathematical physics [6], traffic model [7] and fluid flow [8] and so on.

Scholars and researchers are invited to study books that have been written to better understand the concept of fractional arithmetic [9, 10, 11].

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In order to find the approximate solution for partial differential equations with fractional derivative that we explored in this paper is presented as follows:

$$D_t^\alpha u(x, t) + \mathfrak{A}(x, t, u(x, t), u_x(x, t), u_{xx}(x, t), \dots) = g(x, t), \quad (1.1)$$

in which \mathfrak{A} is the partial differential operator, $D^\alpha u(x, t)$ is the fractional Caputo derivative, $k - 1 < \alpha \leq k$ and $k \in \mathbb{N}$.

A number of articles that can be found to express modeling, deploying and extent of differential equation (DEs), partial differential equation (PDEs) and fractional partial differential equations (FPDEs) are in [12, 13].

It is necessary to announce that there are no accurate analytical solutions for most DEs, PDEs and FPDEs thus; a relatively large number of approximate solution expressed by the scholars are not possible if they find the accurate analytical solutions with the existing procedures for the DEs, PDEs and FPDEs. Accordingly, for such differential equations, we have to employ some direct and iterative methods. Some of these techniques which can be used by scholars include discrete element method and finite difference method [14, 15, 16, 17, 18], homotopy perturbation method (HPM) [19], differential transform method (DTM) [20], Adomian's decomposition method (ADM) [21], optimal homotopy asymptotic method (OHAM) [22], homotopy analysis method (HAM) [23], variational iteration method (VIM) [19], new homotopy asymptotic method (NHPM) [24] and so on [25, 26, 27].

The OHAM was presented and developed by Marinca et al. [28, 29, 30] and it can be shown that HPM is a special case of OHAM. The goal is achieved here by using auxiliary functions, auxiliary convergence controlling parameters, and a homotopy in a particular way to make OHAM simple and effective. The accuracy is also improved with increase in the number of auxiliary parameters in the auxiliary function. Several authors have proved the effectiveness, generalization and reliability of this method. The advantage of OHAM is built in convergence criteria, which is controllable. In OHAM, the control and adjustment of the convergence region are provided in a convenient way. Numerical results show that OHAM is found the best in giving better and more accurate results. It consists of few steps and converges to almost exact solution. The applied method is simple in learning and easy to apply.

This paper is organized as follows: in Section 2, definition and some proposition of the Caputo fractional derivative are introduced. In Section 3, description of OHAM is given. In Section 4, we have expressed the convergence of OHAM. In Section 5, the application

of OHAM to the Eq. 1.1 are illustrated, and some numerical examples are presented. And conclusions are drawn in Section 6.

2. FRACTIONAL CALCULUS

Definition 2.1. A real function $f(x)$, $x > 0$, is considered to be in the space C_ν , ($\nu \in \mathbb{R}$), if there exists a real number $n(> \nu)$, so that $f(x) = x^n f_1(x)$, where $f_1(x) \in C[0, \infty)$, it is said to be in the space C_ν^k if and only if $f^{(k)} \in C_\nu$, $k \in \mathbb{N}$ [10, 11].

Definition 2.2. [10, 11] The Riemann-Liouville fractional integral operator of order $\alpha > 0$, of a function $f \in C_\nu$, $\nu \geq -1$, is given by

$$I_a^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-r)^{\alpha-1} f(r) dr.$$

$$I^\alpha f(x) = I_0^\alpha f(x), \quad I^0 f(x) = f(x).$$

Definition 2.3. [10, 11] The Caputo's fractional derivative of f is defined as

$$D^\alpha f(x) = I^{k-\alpha} D^k f(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-r)^{k-\alpha-1} f^{(k)}(r) dr, \quad x > 0.$$

where, $f \in C_{-1}^k$, $k-1 < \alpha \leq k$ and $k \in \mathbb{N}$.

Proposition 2.1. For $k-1 < \alpha \leq k$, $k \in \mathbb{N}$, $f \in C_\nu^k$, $\nu \geq -1$ and $x > 0$, the following properties satisfy

- i) $D_a^\alpha I_a^\alpha f(x) = f(x)$
- ii) $I_a^\alpha D_a^\alpha f(x) = f(x) - \sum_{j=0}^{k-1} f^{(j)}(a^+) \frac{(x-a)^j}{j!}.$

The Caputo fractional derivative of order α for $u(x, t)$ is defined as:

$$D_t^\alpha u(x, t) = \frac{1}{\Gamma(k+1-\alpha)} \int_0^t (t-s)^{k-\alpha} u^{(k+1)}(x, s) ds, \quad k < \alpha \leq k+1, \quad k \in \mathbb{Z}^+. \quad (2.2)$$

3. DESCRIPTION OF OHAM

The overall dimensions of the proposed approach [31] in this section is given and represented in the following differential equation

$$L\left(u(x, t)\right) + N\left(u(x, t), u(\eta_0(x), \varsigma_0(t)), u_x(\eta_1(x), \varsigma_1(t)), \dots, \underbrace{u_x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t))\right) +$$

$$g(x, t) = 0, \quad x \in \Omega \subseteq \mathbb{R}^n, \quad t > 0 \quad (3.3)$$

featuring the boundary condition

$$B\left(u, \frac{\partial u}{\partial t}\right) = 0, \quad t \in \Gamma, \quad (3.4)$$

in which $L = D_t^\alpha$ is linear operator and N is nonlinear operator may consist of the space derivatives of integer order with respect to x along with delay functions, $u(x, t)$ is unknown function, $g(x, t)$ is a known analytic function, B is a boundary operator, Γ is the boundary of the domain Ω . Also, $\eta_j(x)$ and $\varsigma_j(t)$ are delay functions. In this work, we consider $\eta_j(x) = p_j x$ and $\varsigma_j(t) = q_j t$, for $j = 0, 1, \dots, n$.

According to OHAM, we concoct structural homotopy $v(x, t; p) : \Omega \times [0, 1] \rightarrow \mathbb{R}$ which fulfills the conditions in the following equation

$$(1-p) L(v(x, t; p) - u_0(x, t)) = H(p) \left(L(v(x, t; p)) + g(x, t) + N(u(x, t), u(\eta_0(x), \varsigma_0(t)), u_x(\eta_1(x), \varsigma_1(t)), \dots, \underbrace{u_x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t))) \right), \quad (3.5)$$

where $p \in [0, 1]$ is an embedding parameter, $H(p)$ is a non zero auxiliary function for $p \neq 0$ and $H(0) = 0$. When $p = 0$ and $p = 1$, we have $v(x, t; 0) = u_0(x, t)$ and $v(x, t; 1) = u(x, t)$ respectively.

Thus, when p provides from 0 to 1, the solution $v(x, t; p)$ approaches from the initial guess $u_0(x, t)$ to exact solution $u(x, t)$. In which $u_0(x, t)$ obtained from 3.4 to 3.5 with $p = 0$ giving

$$L(u_0(x, t; 0)) + g(x, t) = 0. \quad (3.6)$$

The auxiliary function $H(p)$ is elected in the following display:

$$H(p) = pc_1 + p^2c_2 + p^3c_3 + \dots, \quad (3.7)$$

in which c_1, c_2, c_3, \dots are convergence control parameters which are unfamiliar and can be calculated. Another demonstration form $H(p)$ offered by Herişanu and his associate in [31]. To compute the approximate solution, we expand $v(x, t; p, c_i)$, in Taylor series around p which is as follows:

$$v(x, t; p, c_i) = u_0(x, t) + \sum_{k=1}^{\infty} u_k(x, t; c_i) p^k, \quad i = 1, 2, \dots \quad (3.8)$$

Defining the vectors

$$\vec{c}_l = \{c_1, c_2, \dots, c_l\}, \quad (3.9)$$

and

$$\begin{aligned}\vec{u}_s = & \{u_0(x, t), u_1(x, t; \vec{c}_1), \dots, u_s(x, t; \vec{c}_s), \\ & (u_0)_x(\eta_1(x), \varsigma_1(t)), (u_1)_x(\eta_1(x), \varsigma_1(t); \vec{c}_1), \dots, (u_s)_x(\eta_1(x), \varsigma_1(t); \vec{c}_s) \\ & \vdots \\ & (u_0)\underbrace{x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t)), (u_1)\underbrace{x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t); \vec{c}_1), \dots, (u_s)\underbrace{x \dots x}_{n \text{ order}}(\eta_n(x), \varsigma_n(t); \vec{c}_s)\}.\end{aligned}$$

The Zero-order problem by (3.6), and the first-order equation by

$$L(u_1(x, t)) = c_1 N_0(\vec{u}_0) + g(x, t) \quad (3.10)$$

and second-order equation by

$$L(u_2(x, t)) - L(u_1(x, t)) = c_2 N_0(\vec{u}_0) + c_1 \left(L(u_1(x, t)) + N_1(\vec{u}_1) \right). \quad (3.11)$$

are considered. The equations in the general case $u_k(x, t)$, are

$$\begin{aligned}L(u_k(x, t)) - L(u_{k-1}(x, t)) = \\ c_k N_0(u_0(x, t)) + \sum_{m=1}^{k-1} c_m \left(L(u_{k-m}(x, t)) + N_{k-m}(\vec{u}_{k-1}) \right),\end{aligned} \quad (3.12)$$

in which $k = 2, 3, \dots$ and N_m is the coefficient of " p^m ", in the development of $N(v(x, t; p))$, about the embedding parameter " p " and we have

$$N(v(x, t; p, c_i)) = N_0(u_0(x, t)) + \sum_{m=1}^{\infty} N_m(\vec{u}_m) p^m. \quad (3.13)$$

It can be seen that, convergence series (3.8) is dependent on the constants c_1, c_2, \dots . If it is convergent at $p = 1$, one has

$$\tilde{v}(x, t; c_i) = u_0(x, t) + \sum_{k=1}^m u_k(x, t; c_i), \quad i = 1, 2, \dots, m. \quad (3.14)$$

The following residual is the result obtained as a result of embedding (3.14) in (3.3):

$$\begin{aligned}R(x, t; c_i) = & L(\tilde{v}(x, t; p, c_i)) + \\ & g(x, t) + N(\tilde{v}(x, t; p, c_i)), \quad i = 1, 2, \dots, m.\end{aligned} \quad (3.15)$$

If $R = 0$, then \tilde{v} will be the exact solution 3.3.

Using the method of least squares and knowing the exact solution of the problem, we can minimize the L^2 -norm of the error $E v_m(c_1, c_2, c_3, \dots, c_m)$. The L^2 -norm of the error is signified as

$$\|E \tilde{v}_m(c_1, \dots, c_m)\|_2 = \left(\int_{\Omega} \int_{\Gamma} \tilde{v}_m^2(x, t) dt dx \right)^{\frac{1}{2}},$$

in which $E \tilde{v}_m(x, t) = |\tilde{v}_{exact}(x, t) - \tilde{v}_m(x, t; c_1, \dots, c_m)|$.

4. CONVERGENCE OF OHAM

Topics in this section are provided for convergence of the OHAM.

Theorem 4.1. [32] *Let the solution components u_0, u_1, u_2, \dots , be defined as given in Eqs.(3.10)-(3.12). The series solution $\sum_{k=0}^{m-1} u_k(x, t)$ defined in 3.14 converges, if $\exists 0 < \rho < 1$ such that $\|u_{k+1}\| \leq \rho \|u_k\| \forall k \geq k_0$ for some $k_0 \in \mathbb{N}$.*

Proof. Under consideration

$$T_0 = u_0$$

$$T_1 = u_0 + u_1$$

$$T_2 = u_0 + u_1 + u_2$$

$$\dots$$

$$T_n = u_0 + u_1 + u_2 + \dots + u_n,$$

as the sequence $\{T_n\}_{n=0}^{\infty}$. Evidence is sufficient to show that the sequence $\{T_n\}_{n=0}^{\infty}$ in the Hilbert space \mathbb{R} is a Cauchy sequence. To achieve this, consider

$$\begin{aligned} \|T_{n+1} - T_n\| &= \|u_{n+1}\| \\ &\leq \rho \|u_n\| \\ &\leq \rho^2 \|u_{n-1}\| \\ &\vdots \\ &\leq \rho^{n-k_0+1} \|u_{k_0}\|. \end{aligned}$$

Assuming that $n \geq m > k_0$ and for every $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
 \|T_n - T_m\| &= \|(T_n - T_{n-1}) + (T_{n-1} - T_{n-2}) + \dots + (T_m - T_{m-1})\| \\
 &\leq \|(T_n - T_{n-1})\| + \|(T_{n-1} - T_{n-2})\| + \dots + \|(T_m - T_{m-1})\| \\
 &\leq \rho^{n-k_0} \|u_{k_0}\| + \rho^{n-k_0-1} \|u_{k_0}\| + \dots + \rho^{m-k_0+1} \|u_{k_0}\| \\
 &= \left(\frac{1 - \rho^{n-m}}{1 - \rho}\right) \rho^{m-k_0+1} \|u_{k_0}\|.
 \end{aligned}$$

According to the $0 < \rho < 1$, it results that $\lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} \|T_n - T_m\| = 0$. Thereupon, in the Hilbert space \mathbb{R} , sequence $\{T_n\}_{n=0}^\infty$ is a Cauchy sequence and this implies that series solution converges to series $\sum_{k=0}^\infty u_k(x, t)$.

5. TEST EXAMPLES

Now that it is easier to understand OHAM, various examples will be described in this section and then will be calculated. These examples include solutions of nonlinear partial differential equation featuring fractional derivative. In all these examples, mathematical software *Mathematica* is used for calculations and graphs.

Example 5.1. *For the first example, we propose the time-fractional advection differential equation:*

$$D_t^\alpha u(x, t) + u(x, t) u_x(x, t) = x(1 + t^2), \quad t > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha \leq 1, \quad (5.16)$$

with the precise solution $u(x, t) = xt$ for $\alpha = 1$ and the primary condition:

$$u(x, 0) = 0. \quad (5.17)$$

Following the OHAM, according to what was formulated and presented in Section 3 for Eqs.(5.16)-(5.17), we get:

$$\begin{aligned}
 u_0(x, t) &= \frac{xt^\alpha (\alpha^2 + 3\alpha + 2t^2 + 2)}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)}, \\
 u_1(x, t) &= -\frac{2c_1 xt^{\alpha+2}}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)} + \frac{2c_1 xt^{\alpha+2}}{(\alpha^3 + 3\alpha^2 + 2\alpha) \Gamma(\alpha)} + \frac{2c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha)} + \\
 &\quad \frac{8c_1 x \Gamma(2\alpha) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 1)} + \frac{13c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \\
 &\quad \frac{12c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{\alpha (\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \frac{\alpha^2 c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \\
 &\quad \frac{24c_1 x \Gamma(2\alpha + 2) t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 4)} + \frac{36c_1 x \Gamma(2\alpha + 3) t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \\
 &\quad \frac{24c_1 x \Gamma(2\alpha + 3) t^{3\alpha+2}}{\alpha (\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \frac{8c_1 x \Gamma(2\alpha + 4) t^{3\alpha+4}}{\alpha (\alpha + 1) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 5)} \\
 &\quad \dots
 \end{aligned}$$

Thereupon, considering the first two sentences as estimates of solution for Eq.(5.16):

TABLE 1. A comparison between approximate solutions with some methods for test example 5.1.

t	x	u_{VIM}	u_{ADM}	u_{HPM}	u_{VHPIM}	u_{Oq-HAM}	u_{OHAM}	u_{Exact}
0.2	0.25	0.050309	0.050000	0.0499876	0.0499876	0.050318	0.050214	0.050000
	0.50	0.100619	0.100000	0.099978	0.0999746	0.091040	0.100428	0.100000
	0.75	0.150928	0.150001	0.149968	0.149962	0.150025	0.150642	0.150000
	1.0	0.201237	0.200001	0.199957	0.199951	0.20100	0.150642	0.200000
0.4	0.25	0.101894	0.100023	0.099645	0.0995290	0.09609	0.101537	0.100000
	0.50	0.203787	0.200046	0.199290	0.199059	0.20370	0.203074	0.200000
	0.75	0.305681	0.300069	0.298935	0.298588	0.300009	0.304611	0.300000
	1.0	0.407575	0.400092	0.398580	0.398118	0.400001	0.304611	0.400000
0.6	0.25	0.153094	0.150411	0.147158	0.145690	0.153001	0.154166	0.150000
	0.50	0.306188	0.300823	0.294317	0.291380	0.300088	0.308331	0.300000
	0.75	0.459282	0.451234	0.441475	0.437070	0.450207	0.462497	0.450000
	1.0	0.612376	0.601646	0.588634	0.582759	0.600633	0.462497	0.600000

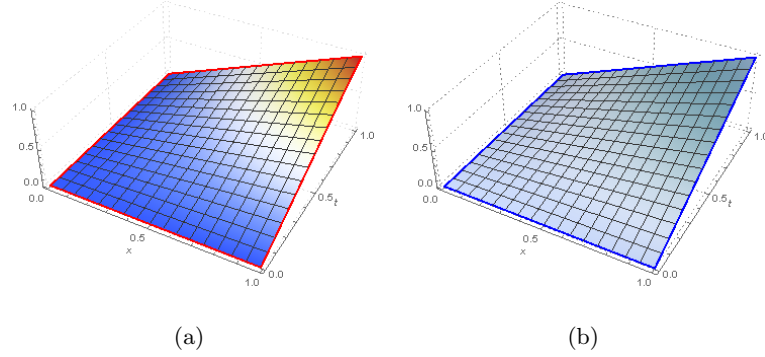


FIGURE 1. (a) The accurate solution (b) The estimate solution in the case $\alpha = 1.0$.

$$\begin{aligned}
 u(x, t) \approx & \frac{xt^\alpha (\alpha^2 + 3\alpha + 2t^2 + 2)}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)} - \frac{2c_1 xt^{\alpha+2}}{\alpha (\alpha^2 + 3\alpha + 2) \Gamma(\alpha)} + \frac{2c_1 xt^{\alpha+2}}{(\alpha^3 + 3\alpha^2 + 2\alpha) \Gamma(\alpha)} + \\
 & \frac{2c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha)} + \frac{8c_1 x \Gamma(2\alpha) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 1)} + \\
 & \frac{13c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \frac{12c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{\alpha (\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \\
 & \frac{\alpha^2 c_1 x \Gamma(2\alpha + 1) t^{3\alpha}}{(\alpha^2 + 3\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 1)} + \frac{24c_1 x \Gamma(2\alpha + 2) t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha)^2 \Gamma(3\alpha + 4)} + \\
 & \frac{36c_1 x \Gamma(2\alpha + 3) t^{3\alpha+2}}{(\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \frac{24c_1 x \Gamma(2\alpha + 3) t^{3\alpha+2}}{\alpha (\alpha + 2)^2 \Gamma(\alpha) \Gamma(\alpha + 2) \Gamma(3\alpha + 4)} + \\
 & \frac{8c_1 x \Gamma(2\alpha + 4) t^{3\alpha+4}}{\alpha (\alpha + 1) \Gamma(\alpha) \Gamma(\alpha + 3) \Gamma(3\alpha + 5)}.
 \end{aligned} \tag{5.18}$$

According to least square method for the calculations of the constants c_1 and c_2 , we can gain $c_1 = 0$, $c_2 = -0.668223$.

In Table 1, we can see the estimated solutions toward $\alpha = 1$, which is derived for various values of x applying OHAM and a comparison between ADM, VIM, HPM, VHPIM and Oq-HAM [7].

In figure 1, we can view the precise and approximate answers featuring $\alpha = 1$.

Table 2 shows comparison between the exact and the approximation solution (5.16) with OHAM of test example 5.1 for different values of α , x and t .

Comparison of exact and approximate solution can be seen for test example 5.1 with different values of α , x and t , in Figure 2.

TABLE 2. The exact and approximate result of test example 5.1 featuring various values of α .

x	t	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 1.0$	u_{Exact}
0.25	0.2	0.114114	0.079887	0.050214	0.05
	0.4	0.148258	0.131658	0.101537	0.1
	0.6	0.164999	0.173966	0.154166	0.15
	0.8	0.162959	0.205729	0.206301	0.2
0.50	0.2	0.228229	0.159774	0.100428	0.1
	0.4	0.296516	0.263317	0.203074	0.2
	0.6	0.329999	0.347933	0.308331	0.3
	0.8	0.325918	0.411458	0.412602	0.4

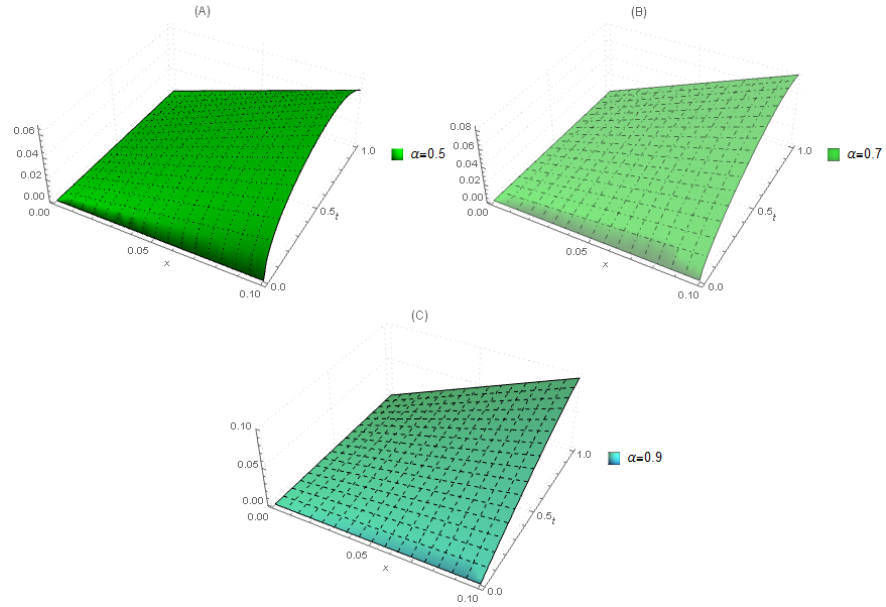


FIGURE 2. Comparison between the exact and the approximation solution with OHAM of test example 5.1 for different values of α , x and t .

Example 5.2. For the second example, we propound the time-fractional Klein-Gordon differential equation:

$$D_t^\alpha u(x, t) - u_{xx}(x, t) + u(x, t) = t^2 + x^2, \quad t > 0, \quad x \in \mathbb{R}, \quad 1 < \alpha \leq 2, \quad (5.19)$$

given that the primary condition

$$u(x, 0) = x^2 - \exp(x), \quad u_t(x, 0) = 0. \quad (5.20)$$

With the help of the OHAM, according to what was formulated and presented in section 3 for Eqs. (5.19)-(5.20), we get:

$$\begin{aligned}
 u_0(x, t) &= x^2 - e^x + \frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} + \frac{x^2 t^\alpha}{\alpha\Gamma(\alpha)}, \\
 u_1(x, t) &= -\frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{x^2 t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \frac{x^2 t^\alpha}{\alpha\Gamma(\alpha)} - \frac{x^2 t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \\
 &\quad \frac{2t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \frac{c_1 x^2 t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \\
 &\quad \frac{c_1 x^2 t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \frac{2c_1 t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2c_1 t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \\
 &\quad \frac{2c_1 t^\alpha}{\alpha\Gamma(\alpha)} - \frac{2c_1 t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \frac{\sqrt{\pi} 2^{1-2\alpha} c_1 t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})} + \\
 &\quad \frac{2c_1 t^{2\alpha+2}}{\Gamma(2\alpha + 3)} + \frac{\sqrt{\pi} 4^{-\alpha} c_1 x^2 t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})}, \\
 &\dots
 \end{aligned}$$

Then, assuming the first two sentences as estimates of solution for Eq.(5.19)

$$\begin{aligned}
 u(x, t) &\approx x^2 - e^x + \frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} + \frac{x^2 t^\alpha}{\alpha\Gamma(\alpha)} - \frac{x^2 t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \frac{x^2 t^\alpha}{\alpha\Gamma(\alpha)} - \frac{x^2 t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \\
 &\quad \frac{2t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \\
 &\quad \frac{c_1 x^2 t^\alpha}{(1-\alpha)\Gamma(\alpha)} - \frac{2c_1 t^\alpha}{\alpha\Gamma(\alpha)} - \frac{c_1 x^2 t^\alpha}{(\alpha-1)\alpha\Gamma(\alpha)} - \frac{2c_1 t^{\alpha+2}}{\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2c_1 t^{2\alpha+2}}{\Gamma(2\alpha + 3)} + \\
 &\quad \frac{2c_1 t^{\alpha+2}}{(\alpha-1)\alpha(\alpha^2 + 3\alpha + 2)\Gamma(\alpha)} - \frac{2c_1 t^{\alpha+2}}{(-\alpha^3 - 2\alpha^2 + \alpha + 2)\Gamma(\alpha)} - \frac{\sqrt{\pi} 2^{1-2\alpha} c_1 t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})} + \\
 &\quad \frac{\sqrt{\pi} 4^{-\alpha} c_1 x^2 t^{2\alpha}}{\alpha\Gamma(\alpha)\Gamma(\alpha + \frac{1}{2})}.
 \end{aligned} \tag{5.21}$$

For the calculations of the constants c_1, c_2 using the method of least squares, we have computed that

$$c_1 = -0.942868, \quad c_2 = 0.00777353.$$

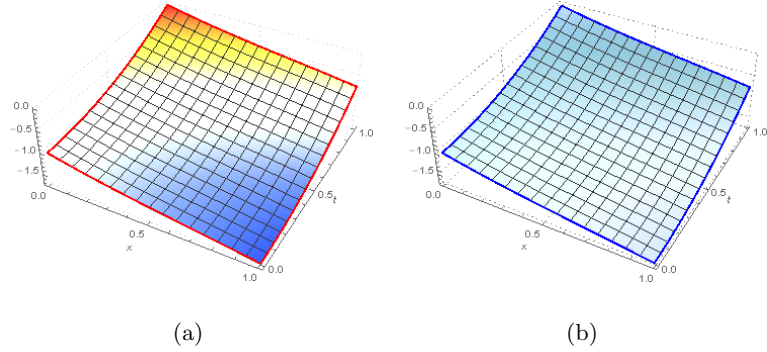
In Table 3 and in figure 3, we can view the precise and approximate answers featuring $\alpha = 2$ through applying OHAM. With the knowledge that $\alpha = 2$, the approximate solution obtained by the proposed method corresponds to the precise solution $u(x, t) = t^2 + x^2 - e^x$.

Example 5.3. For the third example, we offer the time-fractional partial differential equation:

$$D_t^\alpha u(x, t) - u_{xx}(x, t) - u(x, t) = 3t, \quad t > 0, \quad x \in \mathbb{R}, \quad 2 < \alpha \leq 3, \tag{5.22}$$

TABLE 3. Approximate result of test example 5.2.

t	x	u_{OHAM}	$Exact$	$Absolute\ error$
0.0	0.0	-1.	-1.	0.0
0.1	0.5	-0.84694	-0.845171	0.00176917
0.2	0.4	-1.02272	-1.0214	0.00131616
0.3	0.3	-1.17067	-1.16986	0.000814838
0.4	0.2	-1.2922	-1.29182	0.000379601
0.5	0.1	-1.38882	-1.38872	0.0000949325

FIGURE 3. (a) The accurate solution (b) The estimate solution in the case $\alpha = 2.0$.

including the primary condition

$$u(x, 0) = 0, \quad u_t(x, 0) = \sin(x) - 3, \quad u_{tt}(x, 0) = 0. \quad (5.23)$$

With due attention to the OHAM, according to section 3 for Eqs.(5.22)-(5.23), we get:

$$\begin{aligned}
 u_0(x, t) &= \frac{3t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} + t(\sin(x) - 3), \\
 u_1(x, t) &= \frac{3c_1 t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{6c_1 \Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha \Gamma(\alpha)\Gamma(2\alpha + 3)}, \\
 u_2(x, t) &= \frac{3c_1 t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1 t^{2\alpha+1}}{\Gamma(2\alpha + 2)} + \frac{3c_1^2 t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1^2 t^{2\alpha+1}}{\Gamma(2\alpha + 2)} \\
 &\quad - \frac{3c_2 t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} + \frac{6c_1^2 \Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha \Gamma(\alpha)\Gamma(2\alpha + 3)} - \frac{6c_2 \Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha \Gamma(\alpha)\Gamma(2\alpha + 3)} + \frac{3c_1^2 t^{3\alpha+1}}{\Gamma(3\alpha + 2)}. \\
 &\dots
 \end{aligned}$$

Hence, supposing the first two sentences as estimates of solution for Eq.(5.22):

$$\begin{aligned}
 u(x, t) \approx & \frac{3t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} + t(\sin(x) - 3) + \frac{6c_1t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \\
 & \frac{6c_1\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \frac{3c_1^2t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{3c_1^2t^{2\alpha+1}}{\Gamma(2\alpha + 2)} - \frac{6c_1^2\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \\
 & \frac{3c_2t^{\alpha+1}}{(\alpha^2 + \alpha)\Gamma(\alpha)} - \frac{6c_2\Gamma(\alpha + 2)t^{2\alpha+1}}{\alpha\Gamma(\alpha)\Gamma(2\alpha + 3)} + \frac{3c_1^2t^{3\alpha+1}}{\Gamma(3\alpha + 2)}.
 \end{aligned} \tag{5.24}$$

Using the method of least squares, to obtain the constants c_1 and c_2 , we will have

$$c_1 = 0, \quad c_2 = 1.02134.$$

It can be seen in Table 4 and Figure 4 that solving equations with approximate expression is calculated and displayed for $\alpha = 3$ and various values of x and t . Toward $\alpha = 3$, the

TABLE 4. Approximate result of test example 5.3.

t	x	u_{OHAM}	$Exact$	$Absolute\ error$
0.0	0.0	0.0	0.0	0.0
0.1	0.5	-1.45008	-1.45008	0.000161988
0.2	0.4	-1.1206	-1.12053	0.0000672998
0.3	0.3	-0.811365	-0.811344	0.0000214763
0.4	0.2	-0.522121	-0.522116	4.26071×10^{-6}
0.5	0.1	-0.252058	-0.252057	2.6672×10^{-7}

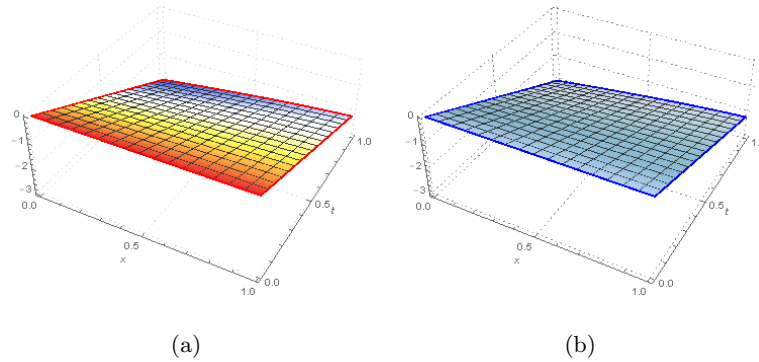


FIGURE 4. (a) The accurate solution (b) The estimate solution in the case $\alpha = 3.0$.

solution that we have gained is in accordance with the precise solution $u(x, t) = t \sin(x) - 3t$.

6. CONCLUSION

We have successfully applied OHAM to obtain approximate solution of the non linear partial differential equations featuring fractional derivative. The result indicate that a few iteration of OHAM will results in some useful solutions.

Finally, it should be added that the suggested technique has the potentials to be practical in solving other similar nonlinear and linear problems in partial differential equations featuring fractional derivative.

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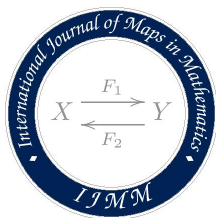
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ALMOST G-CONTACT METRIC MANIFOLD

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ABSTRACT. In this paper, starting from only a global basis of vector fields, we construct a class of almost contact metric manifolds and we give concrete example. Next, we study some essential types belonging to this class on dimension 3 and we construct several examples.

1. INTRODUCTION

Fortunately, the rich theory of vector spaces endowed with a Euclidean inner product can, to a great extent, be lifted to various bundles associated with a manifold. The notion of global (and local) frame plays an important technical role.

It should be mentioned however that a global basis of $\mathfrak{X}(M)$ (the Lie algebra of smooth vector fields on a manifold M) i.e., n vector fields that are linearly independent over $\mathcal{F}(M)$ and span $\mathfrak{X}(M)$, does not exist in general.

Manifolds that do admit such a global basis for $\mathfrak{X}(M)$ are called parallelizable. it is straightforward to show that a finite-dimensional manifold is parallelizable if and only if its tangent bundle is trivial (that is, isomorphic to the product, $M \times \mathbb{R}^n$).

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As an illustration, we can prove that the tangent bundle, TS^1 , of the circle, is trivial. Indeed, we can find a section that is everywhere nonzero, i.e. a non-vanishing vector field, namely

$$X(\cos\theta, \sin\theta) = (-\sin\theta, \cos\theta).$$

The reader should try proving that TS^3 is also trivial (use the quaternions). However, TS^2 is nontrivial, although this not so easy to prove.

More generally, it can be shown that TS^n is nontrivial for all even $n \geq 2$. It can even be shown that S^1, S^3 and S^7 are the only spheres whose tangent bundle is trivial. This is a rather deep theorem and its proof is hard.

Here, starting from a Global frame we construct a class of almost contact metric structures, specifically, many well-known almost contact metric structures (Sasakian, cosymplectic, Kenmotsu) in dimension three and we confirm the construction each time with a concrete example showing that the case is non-vacuous.

This work is organized in the following way:

Section 2 is devoted to the background of the structures which will be used in the sequel. In **Section 3**, we give the necessary techniques to construct an almost contact metric structure from a global frame of vector fields and we give an example. In **Section 4**, we focus on the case of three-dimensional and we show how to construct some basic structures with concrete examples.

2. REVIEW OF NEEDED NOTIONS

An odd-dimensional Riemannian manifold (M^{2n+1}, g) is said to be an almost contact metric manifold if there exist on M a $(1, 1)$ -tensor field φ , a vector field ξ (called the structure vector field) and a 1-form η such that

$$\left\{ \begin{array}{l} (1) : \eta(\xi) = 1, \\ (2) : \varphi^2(X) = -X + \eta(X)\xi, \\ (3) : g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y). \end{array} \right. \quad (2.1)$$

for any vector fields X, Y on M . In particular, in an almost contact metric manifold we also have

$$\varphi\xi = 0 \quad \text{and} \quad \eta \circ \varphi = 0. \quad (2.2)$$

Such a manifold is said to be a contact metric manifold if $d\eta = \Omega$, where $\Omega(X, Y) = g(X, \varphi Y)$ is called the fundamental 2-form of M . If, in addition, ξ is a Killing vector field,

then M is said to be a K-contact manifold. It is well-known that a contact metric manifold is a K-contact manifold if and only if $\nabla_X \xi = -\varphi X$, for any vector field X on M .

On the other hand, the almost contact metric structure of M is said to be normal if

$$N_\varphi(X, Y) = [\varphi, \varphi](X, Y) + 2d\eta(X, Y)\xi = 0, \quad (2.3)$$

for any X, Y , where $[\varphi, \varphi]$ denotes the Nijenhuis torsion of φ , given by

$$[\varphi, \varphi](X, Y) = \varphi^2[X, Y] + [\varphi X, \varphi Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$$

An almost contact metric structures (φ, ξ, η, g) on M is said to be:

$$\left\{ \begin{array}{l} (a) : \textit{Sasaki} \Leftrightarrow \Omega = d\eta \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\ (b) : \textit{Cosymplectic} \Leftrightarrow d\Omega = d\eta = 0 \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\ (c) : \textit{Kenmotsu} \Leftrightarrow d\eta = 0, d\Omega = 2\eta \wedge \Omega \text{ and } (\varphi, \xi, \eta) \text{ is normal.} \end{array} \right. \quad (2.4)$$

where d denotes the exterior derivative.

These manifolds can be characterized through their Levi-Civita connection, by requiring

$$\left\{ \begin{array}{l} (1) : \textit{Sasaki} \Leftrightarrow (\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X, \\ (2) : \textit{Cosymplectic} \Leftrightarrow \nabla \varphi = 0, \\ (3) : \textit{Kenmotsu} \Leftrightarrow (\nabla_X \varphi)Y = g(\varphi X, Y)\xi - \eta(Y)\varphi X. \end{array} \right. \quad (2.5)$$

For more background on almost contact metric manifolds, we recommend the reference [1],[2], [3] and [5].

3. ALMOST G-CONTACT METRIC MANIFOLD

Let $\{e_0, e_i\}_{1 \leq i \leq 2n}$ be the global frame of vector fields and $\{\theta^0, \theta^i\}_{1 \leq i \leq 2n}$ be the dual frame of differential 1-forms on a $(2n+1)$ -dimensional smooth manifold M . Define a $(1, 1)$ -tensor field φ on M by

$$\varphi = \sum_{i=1}^n e_{2i} \wedge e_{2i-1} = \sum_{i=1}^n (\theta^{2i-1} \otimes e_{2i} - \theta^{2i} \otimes e_{2i-1}), \quad (3.6)$$

i.e. for all vector field X on M , we have

$$\begin{aligned} \varphi X &= \sum_{i=1}^n (e_{2i} \wedge e_{2i-1})X \\ &= \sum_{i=1}^n (g(e_{2i-1}, X)e_{2i} - g(e_{2i}, X)e_{2i-1}) \\ &= \sum_{i=1}^n \theta^{2i-1}(X)e_{2i} - \theta^{2i}(X)e_{2i-1}, \end{aligned}$$

and a Riemannian metric g on M with $\{e_i\}_{0 \leq i \leq 2n+1}$ is an orthonormal frame, so that

$$g = \sum_{i=0}^{2n} \theta^i \otimes \theta^i. \quad (3.7)$$

With these identities, we state the following:

Theorem 3.1. *The manifold $(M, \varphi, e_0, \theta^0, g)$ defined as above is an almost contact metric manifold.*

Proof. According to the conditions (2.1), the data $(M, g, \varphi, e_0, \theta^0, g)$ is an almost contact metric manifold if only two conditions are satisfied

$$\varphi^2 X = -X + \theta^0(X)e_0 \quad \text{and} \quad g(\varphi X, \varphi Y) = g(X, Y) - \theta^0(X)\theta^0(Y).$$

Using formula (3.6) we get

$$\varphi e_{2i} = -e_{2i-1} \quad \text{and} \quad \varphi e_{2i-1} = e_{2i}.$$

To prove the first condition, we have for all X vectors field on M

$$\begin{aligned} \varphi^2 X &= \sum_{i=1}^n (\theta_{2i-1}(X)\varphi e_{2i} - \theta^{2i}(X)\varphi e_{2i-1}) \\ &= -\sum_{i=1}^n (\theta^{2i-1}(X)e_{2i-1} + \theta^{2i}(X)e_{2i}) \\ &= -\sum_{i=1}^{2n} \theta^i(X)e_i \\ &= -X + \theta^0(X)e_0. \end{aligned}$$

For the second condition, for all X and Y vectors fields on M we have

$$\begin{aligned} g(\varphi X, \varphi Y) &= \sum_{i=1}^n (\theta^{2i-1}(X)\theta^{2i-1}(Y) + \theta^{2i}(X)\theta^{2i}(Y)) \\ &= \sum_{i=1}^{2n} \theta^i(X)\theta^i(Y) \\ &= g\left(X, \sum_{i=1}^{2n} \theta^i(Y)e_i\right) \\ &= g(X, Y - \theta^0(Y)e_0) \\ &= g(X, Y) - \theta^0(X)\theta^0(Y), \end{aligned}$$

which completes the proof.

We refer to this construction as **almost G-contact metric manifold**.

Example 3.1. Let (x_i) be the Cartesian coordinates in \mathbb{R}^5 and $\partial_i = \frac{\partial}{\partial x_i}$. Define a global frame of vector fields on \mathbb{R}^5 by:

$$e_0 = \partial_5, \quad e_1 = \partial_1 + f\partial_5, \quad e_2 = \partial_2, \quad e_3 = \partial_3 + h\partial_5, \quad e_4 = \partial_4,$$

where f, h are two strictly positive functions on \mathbb{R}^5 and let g be the Riemannian metric defined by

$$g(e_i, e_j) = \delta_{ij} \quad \forall i, j \in \{0, \dots, 5\},$$

that is, the form of the metric becomes

$$g = \begin{pmatrix} 1+f^2 & 0 & fh & 0 & -f \\ 0 & 1 & 0 & 0 & 0 \\ fh & 0 & 1+h^2 & 0 & -h \\ 0 & 0 & 0 & 1 & 0 \\ -f & 0 & -h & 0 & 1 \end{pmatrix},$$

and the 1-form corresponding to e_0 is $\theta^0 = -f dx_1 - h dx_3 + dx_5$.

To define φ , let us use the formula

$$\begin{aligned} \varphi &= \sum_{i=1}^2 e_{2i} \wedge e_{2i-1} \\ &= e_2 \wedge e_1 + e_4 \wedge e_3 \\ &= \theta^1 \otimes e_2 - \theta^2 \otimes e_1 + \theta^3 \otimes e_4 - \theta^4 \otimes e_3, \end{aligned}$$

we get

$$\varphi = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -f & 0 & -h & 0 \end{pmatrix},$$

where we can check that $(\mathbb{R}^5, \varphi, e_0, \theta^0, g)$ is an almost G-contact metric manifold.

Remark 3.1. Any almost G-contact metric manifold is an almost contact metric manifold, the converse is not true in general.

While this is an area of possible future research we mention briefly that one easily has the following:

The fundamental 2-form Φ of $(\varphi, e_0, \theta^0, g)$ is :

$$\begin{aligned}
 \Phi(X, Y) &= g(X, \varphi Y) \\
 &= g\left(X, \sum_{i=1}^n (e_{2i} \wedge e_{2i-1})Y\right) \\
 &= g\left(X, \sum_{i=1}^n \theta^{2i-1}(Y)e_{2i} - \theta^{2i}(Y)e_{2i-1}\right) \\
 &= \sum_{i=1}^n \left(\theta^{2i-1}(Y)\theta^{2i}(X) - \theta^{2i}(Y)\theta^{2i-1}(X)\right) \\
 &= 2 \sum_{i=1}^n (\theta^{2i} \wedge \theta^{2i-1})(X, Y),
 \end{aligned}$$

we can check that is very simply as follows:

$$\Phi = 2 \theta^{2i} \wedge \theta^{2i-1}. \quad (3.8)$$

Proposition 3.1. *Let $(M, \Phi, e_0, \theta^0, g)$ be an almost G -contact manifold. Then, we have*

$$d\Phi = \frac{\alpha}{n} \theta^0 \wedge \Phi, \quad (3.9)$$

where d denote the exterior derivative and

$$\alpha = \text{div} e_0 + \sum_{i < j} \left((\mathcal{L}_{e_0} g)(e_{2i-1}, e_{2j-1}) + (\mathcal{L}_{e_0} g)(e_{2i}, e_{2j}) \right).$$

Proof. Let $U = \sum_{i=1}^n e_{2i-1}$ and $V = \sum_{i=1}^n e_{2i}$ two vectors fields on M . Putting $d\Phi = \sigma \theta^0 \wedge \Phi$ for a certain functions σ on M . Then, we get

$$3 (\theta^0 \wedge \Phi)(e_0, U, V) = n,$$

$$\begin{aligned}
 3d\Phi(e_0, U, V) &= (\nabla_{e_0} \Phi)(U, V) + (\nabla_U \Phi)(V, \xi) + (\nabla_V \Phi)(\xi, U) \\
 &= -\Phi(V, \nabla_U \theta^0) - \Phi(\nabla_V \theta^0, U) \\
 &= \sum_{i=1}^n (\theta^{2i-1}(\nabla_U \theta^0) + \theta^{2i}(\nabla_V \theta^0)) \\
 &= g(\nabla_U \theta^0, U) + g(\nabla_V \theta^0, V) \\
 &= \text{div} \xi + \sum_{i < j} \left((\mathcal{L}_{e_0} g)(e_{2i-1}, e_{2j-1}) + (\mathcal{L}_{e_0} g)(e_{2i}, e_{2j}) \right) \\
 &= \alpha,
 \end{aligned}$$

which implies $\sigma = \frac{\alpha}{n}$.

4. 3-DIMENSIONAL ALMOST G-CONTACT METRIC MANIFOLD

Let $\{e_0, e_1, e_2\}$ be the global frame of vector fields and $\{\theta^0, \theta^1, \theta^2\}$ be the dual frame of differential 1-forms on a 3-dimensional smooth manifold M^3 . Define a $(1, 1)$ -tensor field φ on M by

$$\varphi = e_2 \wedge e_1 = \theta^1 \otimes e_2 - \theta^2 \otimes e_1 \quad (4.10)$$

and a Riemannian metric g on M with $\{e_i\}_{0 \leq i \leq 2}$ is an orthonormal frame, so that

$$g = \sum_{i=0}^2 \theta^i \otimes \theta^i. \quad (4.11)$$

According to the theorem 3.1, $(M^3, \varphi, e_0, \theta^0, g)$ is an almost G-contact metric manifold.

Through the rest of this paper, we are mainly interested in dimension three. Below we recall certain results concerning this case.

For an arbitrary 3-dimensional almost contact metric manifold (M^3, ξ, η, g) , we have

$$d\Phi = 2\alpha\eta \wedge \Phi. \quad (4.12)$$

A 3-dimensional almost contact metric manifold M is normal if and only if for all X vectors field on M ([4], Prop. 2)

$$\nabla_{\varphi X} \xi = \varphi \nabla_X \xi, \quad (4.13)$$

or, equivalently,

$$\nabla_X \xi = -\alpha\varphi^2 X - \beta\varphi X, \quad (4.14)$$

and for a normal almost contact metric manifold M we have ([4], Corollary 1)

$$\nabla_{\xi} \xi = 0 \quad \text{and} \quad d\eta = \beta\Phi. \quad (4.15)$$

where α and β are the functions defined by $2\alpha = \text{div} \xi$ and $2\beta = \text{tr}(\varphi \nabla \xi)$ and ∇ is the Levi-Civita connection on M .

From formulas (2.4) and (4.12)-(4.15), one can easily proof that

$$\left\{ \begin{array}{l} (a) : \text{Sasaki} \Leftrightarrow \alpha = 0, \beta = 1 \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\ (b) : \text{Cosymplectic} \Leftrightarrow \alpha = \beta = 0 \text{ and } (\varphi, \xi, \eta) \text{ is normal,} \\ (c) : \text{Kenmotsu} \Leftrightarrow \alpha = 1, \beta = 0 \text{ and } (\varphi, \xi, \eta) \text{ is normal.} \end{array} \right. \quad (4.16)$$

As a consequence of the above formulas (2.5), we immediately obtain the following result:

Theorem 4.1. *A 3-dimensional almost G-contact metric manifold is:*

$$\left\{ \begin{array}{l} (1) : G - \text{Sasaki} \Leftrightarrow \nabla_X e_0 = -\varphi X, \\ (2) : G - \text{cosymplectic} \Leftrightarrow \nabla_X e_0 = 0, \\ (3) : G - \text{Kenmotsu} \Leftrightarrow \nabla_X e_0 = -\varphi^2 X. \end{array} \right. \quad (4.17)$$

for all vectors field X on M .

Proof. According to the cases given in formulas (2.5) we have:

(1): An almost G-contact metric manifold is Sasakian if and only if

$$(\nabla_X \varphi)Y = g(X, Y)e_0 - \theta^0(Y)X, \quad (4.18)$$

taking $Y = e_0$ with $\theta^0(\nabla_X e_0) = 0$, we obtain

$$\begin{aligned} (\nabla_X \varphi)e_0 = \theta^0(X)e_0 - X &\Leftrightarrow -\varphi \nabla_X e_0 = \theta^0(X)e_0 - X \\ &\Leftrightarrow -\varphi^2 \nabla_X e_0 = -\varphi X \\ &\Leftrightarrow \nabla_X e_0 = -\varphi X, \end{aligned}$$

we proved that if M is G-Sasakian then $\nabla_X e_0 = -\varphi X$. Conversely, suppose that

$$\nabla_X e_0 = -\varphi X. \quad (4.19)$$

It is easy to see that $\nabla_{\varphi X} e_0 = \varphi \nabla_X e_0$, then the structure (φ, e_0, θ^0) is normal and also we have (see [4], Prop. 2)

$$\nabla_X e_0 = -\alpha \varphi^2 X - \beta \varphi X. \quad (4.20)$$

From formulas (4.19) and (4.20), we get

$$\alpha = 0 \quad \text{and} \quad \beta = 1,$$

following formulas (4.16), M is a G-Sasakian manifold.

(2): An almost G-contact metric manifold is cosymplectic if and only if

$$(\nabla_X \varphi)Y = 0, \quad (4.21)$$

taking $Y = e_0$, we obtain $\nabla_X e_0 = 0$.

Conversely, suppose that

$$\nabla_X e_0 = 0. \quad (4.22)$$

It is easy to see that $\nabla_{\varphi X} e_0 = \varphi \nabla_X e_0 = 0$, then the structure (φ, e_0, θ^0) is normal, using formulas (4.14), we get

$$\alpha = \beta = 0,$$

following formulas (4.16), M is a G-cosymplectic manifold.

(3): An almost G-contact metric manifold is Kenmotsu if and only if

$$(\nabla_X \varphi)Y = g(\varphi X, Y)e_0 - \theta^0(Y)\varphi X, \quad (4.23)$$

taking $Y = e_0$, we get $\nabla_X e_0 = -\varphi^2 X$. we proved that if M is G-Kenmotsu then $\nabla_X e_0 = -\varphi^2 X$. Conversely, suppose that

$$\nabla_X e_0 = -\varphi^2 X. \quad (4.24)$$

we obtain

$$\nabla_{\varphi X} e_0 = -\varphi^3 X = \varphi \nabla_X e_0,$$

therefore, the structure (φ, e_0, θ^0) is normal and also we have (see [4], Prop. 2)

$$\nabla_X e_0 = -\alpha \varphi^2 X - \beta \varphi X. \quad (4.25)$$

From formulas (4.24) and (4.25), we get

$$\alpha = 1 \quad \text{and} \quad \beta = 0,$$

following formulas (4.16), M is a G-Kenmotsu manifold.

5. EXAMPLES

Let (x, y, z) denote the Cartesian coordinates in \mathbb{R}^3 . We denote the global frame of vector fields on \mathbb{R}^3 by (e_0, e_1, e_2) and the dual frame of differential 1-forms by $(\theta^0, \theta^1, \theta^2)$ such that $\theta^i(e_j) = \delta_{ij}$ for all $i, j \in \{0, 1, 2\}$.

Example 5.1. (*G-Sasakian manifold*)

Consider

$$\theta^0 = dx + 2zdy, \quad \theta^1 = dy, \quad \theta^2 = dz,$$

and

$$e_0 = \frac{\partial}{\partial x}, \quad e_1 = -2z \frac{\partial}{\partial x} + \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z}.$$

For the non-zero Lie brackets of (e_i) , we have:

$$[e_1, e_2] = 2e_0.$$

Define an almost contact structure (φ, e_0, θ^0) on M by assuming

$$\varphi e_0 = 0, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1.$$

Let g be the Riemannian metric on M for which (e_i) is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, e_0, \theta^0, g)$ is an almost contact metric structure on \mathbb{R}^3 . For the Levi-Civita connection corresponding to g , we have

$$\nabla_{e_0} e_0 = \nabla_{e_1} e_1 = \nabla_{e_2} e_2 = 0, \quad \nabla_{e_0} e_1 = \nabla_{e_1} e_0 = -e_2,$$

$$\nabla_{e_0} e_2 = \nabla_{e_2} e_0 = e_1, \quad \nabla_{e_1} e_2 = -\nabla_{e_2} e_1 = e_0.$$

We can easily check that $i \in \{0, 1, 2\}$

$$\nabla_{e_i} e_0 = -\varphi e_i.$$

Knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$ for all X and Y vectors fields on M , one can check that

$$(\nabla_{e_i} \varphi)e_j = \delta_{ij} e_0 - \theta^0(e_j) e_i,$$

for all $i, j \in \{0, 1, 2\}$. Therefore, $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G -Sasakian manifold.

Example 5.2. (G -cosymplectic manifold)

For the global frame

$$e_0 = \frac{\partial}{\partial x} + x \frac{\partial}{\partial z}, \quad e_1 = \frac{\partial}{\partial y}, \quad e_2 = \frac{\partial}{\partial z},$$

we define a Riemannian metric g by

$$g(e_0, e_1) = g(e_0, e_2) = g(e_1, e_2) = 0,$$

$$g(e_0, e_0) = g(e_1, e_1) = g(e_2, e_2) = 1$$

that is, the form of the metric becomes

$$g = \begin{pmatrix} 1+x^2 & 0 & -x \\ 0 & 1 & 0 \\ -x & 0 & 1 \end{pmatrix},$$

and the corresponding 1-forms are

$$\theta^0 = dx, \quad \theta^1 = dy, \quad \theta^2 = -x dx + dz,$$

To define φ , let's use the formula $\varphi = e_2 \wedge e_1$, we get

$$\varphi = \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

where we can check that (φ, ξ, η, g) is an almost G -contact metric structure on E^3 .

It is easy to see that for all $i, j \in \{0, 1, 2\}$,

$$[e_i, e_j] = 0,$$

therefore, all components of the Levi-Civita connection are zero. Then, for all $i \in \{0, 1, 2\}$ we obtain

$$\nabla_{e_i} e_0 = 0,$$

which shows that $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G -cosymplectic manifold. One can verify this result by classical reasoning, using formulas (4.16).

Example 5.3. (G -Kenmotsu manifold)

Consider

$$\theta^0 = -x dx + dz, \quad \theta^1 = e^z dx, \quad \theta^2 = e^z dy,$$

and

$$e_0 = \frac{\partial}{\partial z}, \quad e_1 = e^{-z} \left(\frac{\partial}{\partial x} + x \frac{\partial}{\partial z} \right), \quad e_2 = e^{-z} \frac{\partial}{\partial y}.$$

For the non-zero Lie brackets of (e_i) , we have:

$$[e_0, e_1] = -e_1, \quad [e_0, e_2] = -e_2, \quad [e_1, e_2] = -x e^{-z} e_2.$$

Define an almost contact structure (φ, e_0, θ^0) on M by assuming

$$\varphi e_0 = 0, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -e_1.$$

Let g be the Riemannian metric on M for which (e_i) is an orthonormal frame, so that $g = \sum \theta^i \otimes \theta^i$. It is obvious that $(\varphi, e_0, \theta^0, g)$ is an almost contact metric structure on \mathbb{R}^3 .

For the Levi-Civita connection corresponding to g , we have

$$\nabla_{e_0} e_0 = \nabla_{e_0} e_1 = \nabla_{e_0} e_2 = \nabla_{e_1} e_2 = 0,$$

$$\nabla_{e_1} e_0 = e_1, \quad \nabla_{e_2} e_0 = e_2, \quad \nabla_{e_1} e_1 = e_0$$

$$\nabla_{e_2} e_1 = x e^{-z} e_2, \quad \nabla_{e_2} e_2 = -e_0 - x e^{-z} e_1.$$

We can see that for all $i \in \{0, 1, 2\}$

$$\nabla_{e_i} e_0 = -\varphi^2 e_i.$$

Knowing that $(\nabla_X \varphi)Y = \nabla_X \varphi Y - \varphi \nabla_X Y$, one can check that

$$(\nabla_X \varphi)Y = g(\varphi X, Y)e_0 - \theta^0(Y)\varphi X,$$

for all $X, Y \in \{e_0, e_1, e_2\}$. Therefore, $(\mathbb{R}^3, \varphi, e_0, \theta^0, g)$ is a G -Kenmotsu manifold.

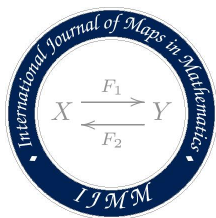
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NULL HYPERSURFACES IN INDEFINITE NEARLY KAEHLERIAN FINSLER SPACE-FORMS

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ABSTRACT. We study the geometry of null hypersurfaces, M , in indefinite nearly Kaehlerian Finsler space-forms \mathbb{F}^{2n} . We prove new inequalities involving the point-wise vertical sectional curvatures of \mathbb{F}^{2n} , based on two special vector fields on an umbilic hypersurface. Such inequalities generalize some known results on null hypersurfaces of Kaehlerian space forms. Furthermore, under some geometric conditions, we show that the null hypersurface (M, B) , where B is the local second fundamental form of M , is locally isometric to the null product $M_D \times M_{D'}$, where M_D and $M_{D'}$ are the leaves of the distributions D and D' which constitutes the natural null-CR structure on M .

1. INTRODUCTION

A Finsler manifold is a manifold \mathbb{F} where each tangent space is equipped with a Minkowski norm, that is, a norm that is not necessarily induced by an inner product. This norm also induces a canonical inner product. However, in sharp contrast to the Riemannian case, these Finsler-inner products are not parameterized by points of \mathbb{F} , but by directions in the tangent space of \mathbb{F} . Thus one can think of a Finsler manifold as a space where the inner product does not only depend on where you are, but also in which direction you are looking. For example,

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length, geodesics, curvature, connections, covariant derivative, and structure equations all generalize. However, normal coordinates do not [22]. More details on the basics of Finsler spaces can be found in [17, 22], and any other references cited therein. Subspaces of definite Finsler spaces have been investigated in details. For example, [12] has studied the geometry of CR-submanifolds of Kaehlerian Finsler spaces.

Indefinite Finsler spaces have also been studied by many researchers, for instance see [8, 9]. Null subspaces naturally exists in indefinite spaces, and in case of semi-Riemannian spaces, they have been investigated to a good depth by a lot of scholars like [1, 2, 7, 11, 14, 15, 16, 18, 19, 21]. Despite such numerous work on null subspaces of indefinite semi-Riemannian spaces, there is only one paper by A. Bejancu [4] which talks about null hypersurfaces of indefinite Finsler spaces. The paper lays out the geometric objects induced on such hypersurfaces and also discusses the structure equations involving the vertical curvature tensors. The aim of this paper is to extend his work by fully investing the geometry of null hypersurfaces (M, g) in indefinite nearly Kaehlerian Finsler space-forms. Several new classification results are proved on totally umbilic hypersurfaces, as well as the geometry of the null hypersurface (M, B) , where M is the second fundamental form of M . Such a hypersurface has also been studied by Bejan and Duggal in [7]. The paper is arranged as follows; Section 2 focusses on the basic preliminaries on Finsler spaces as well null subspaces necessary for the rest of the paper. In Section 3, we discuss totally umbilic hypersurfaces in which we give conditions on the sectional curvature of \mathbb{F} depending on two vector fields U and V on M , as well as its null sectional curvature (see Theorems 3.1 and 3.2). Finally, Section 4 is devoted to the geometry of (M, B) in which we show that its a product manifold under some geometric conditions (see Theorems 4.2 and 4.3).

2. PRELIMINARIES

Let \overline{M}^{2n} be a smooth $2n$ -dimensional manifold and $T\overline{M}^{2n}$ be the tangent bundle of \overline{M} . Let $i : \overline{M}^{2n} \rightarrow T\overline{M}^{2n}$ be the natural imbedding, i.e., $i(x) = 0_x \in T_x\overline{M}^{2n}$, for $x \in \overline{M}^{2n}$. Let us put $T\overline{M}' = T\overline{M}^{2n} \setminus i(\overline{M}^{2n})$. The coordinates of a point of $T\overline{M}^{2n}$ are denoted by (x^i, y^i) , where (x^i) and (y^i) are the coordinates of a point $x \in \overline{M}^{2n}$ and the components of a vector $y \in T_x\overline{M}^{2n}$, respectively. Consider a continuous function $L(x, y)$, for $(x, y) \in T\overline{M}'$, defined on $T\overline{M}^{2n}$ and suppose that the following conditions are satisfied

- (1) L is smooth on $T\overline{M}'$.
- (2) $L(x, \lambda y) = \lambda L(x, y)$, for all $\lambda \in [0, \infty)$.

(3) The metric tensor

$$\bar{g}_{ij}(x, y) = \frac{1}{2} \frac{\partial^2 L^2}{\partial y^i \partial y^j}, \quad (2.1)$$

is positive definite.

Then, we say that (\bar{M}^{2n}, L) is a Finsler manifold [22].

Let $\pi : T\bar{M}' \longrightarrow \bar{M}^{2n}$ be the natural projection and $\pi^{-1}T\bar{M}^{2n} \longrightarrow T\bar{M}'$ be the pullback bundle of $T\bar{M}^{2n}$ by π . A bundle morphism $\bar{J} : \pi^{-1}T\bar{M}^{2n} \longrightarrow \pi^{-1}T\bar{M}$, $\bar{J}^2 = -I$ is said to be a *Finslerian almost complex structure* [12] on \bar{M}^{2n} . Let $u \in T\bar{M}'$, then $\pi_u^{-1}T\bar{M}^{2n} = \{u\} \times T_x\bar{M}^{2n}$, $x = \pi(u)$, denotes the fiber over u in $\pi^{-1}T\bar{M}^{2n}$. Moreover, any ordinary almost complex structure $\bar{J} : T\bar{M}^{2n} \longrightarrow T\bar{M}^{2n}$, $\bar{J}^2 = -I$, admits a natural lift to the Finslerian almost complex structure \tilde{J} given by $\tilde{J}_u X = (u, \bar{J}_x \hat{\pi} X)$, $x = \pi(u)$, $X \in \pi_u^{-1}T\bar{M}^{2n}$, $u \in T\bar{M}'$, where $\hat{\pi}$ denotes the projection onto the second factor of $T\bar{M}' \times T\bar{M}^{2n}$. Denote by $\mathcal{V}T\bar{M}'$ the *vertical vector bundle* over $T\bar{M}'$, that is, $\mathcal{V}T\bar{M}' = \ker d\pi$, where $d\pi$ is the differential of π . Then, any section of $\mathcal{V}T\bar{M}'$ is called a Finsler vector field. Also, any section of the dual vector bundle $\mathcal{V}^*T\bar{M}'$ is a Finsler 1-form.

Let \mathbb{F}^{2n} be a Finsler space endowed with the Finslerian almost complex structure \bar{J} . Then, \mathbb{F}^{2n} is said to be an *almost Hermitian Finsler space* [12] if

$$\bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}(X, Y), \quad (2.2)$$

for all $X, Y \in \Gamma(\mathcal{V}T\bar{M}')$. A connection $\bar{\nabla}$ in the induced bundle $(\pi^{-1}T\bar{M}, \bar{g})$ is called *metrical* [12] (resp. *almost complex*) if

$$\bar{\nabla} \bar{g} = 0, \quad \text{resp.} \quad \bar{\nabla} \bar{J} = 0. \quad (2.3)$$

A tangent vector Z on $T\bar{M}'$ is called horizontal if $\bar{\nabla}_Z v = 0$, where v is the *Liouville vector field*. Let \bar{N} be the distribution of all horizontal vector fields of $T\bar{M}'$. It is referred to as the *horizontal distribution* [12] of $\bar{\nabla}$. Then $\bar{\nabla}$ is *regular* if its horizontal distribution is *nonlinear connection* on $T\bar{M}'$. The pair $(\bar{N}, \bar{\nabla})$ consisting of a connection in $\pi^{-1}T\bar{M}^{2n}$ and a nonlinear connection on $T\bar{M}'$ is called a Finsler connection on \bar{M}^{2n} . The fundamental theorem of Finsler geometry asserts that there exists a regular connection $\bar{\nabla}$ in the induced bundle $(\pi^{-1}T\bar{M}, \bar{g})$, called a *Cartan connection* [12] on (\bar{M}^{2n}, L) . Consequently, $(\bar{M}^{2n}, L, \bar{J})$ is called a *Kaehlerian Finsler space* if its Cartan connection is almost complex (see [12] for more details).

According to Bejancu [3], an almost Hermitian manifold \overline{M} is nearly Kaehlerian if its Levi-Civita connection $\overline{\nabla}$ satisfies the relation

$$(\overline{\nabla}_X \overline{J})Y + (\overline{\nabla}_Y \overline{J})X = 0, \quad (2.4)$$

for any $X, Y \in \Gamma(T\overline{M})$. It then follows directly that a Kaehlerian manifold is a nearly Kaehlerian manifold. Let $(\overline{M}^{2n}, L, \overline{J})$ be an almost Hermitian Finsler space. We say that $(\overline{M}^{2n}, L, \overline{J})$ is a *nearly Kaehlerian Finsler space* if its Cartan connection $\overline{\nabla}$ satisfies (2.4).

Let $\overline{\nabla}$ be the Cartan connection of the Finsler space (\overline{M}^{2n}, L) and $\overline{R}^\mathcal{V}$ its vertical curvature tensor. If $u \in T\overline{M}'$ and p is a 2-dimensional real subspace of a fibre $\pi_u^{-1}\overline{M}^{2n}$, let $s(p) = \overline{R}_u^\mathcal{V}(X, Y, X, Y)$, for some \overline{g}_u -orthonormal basis $\{X, Y\}$ in p , be the *vertical sectional curvature* of (\overline{M}^{2n}, L) [13, p. 97]. Let $\sigma : GF_2(\overline{M}^{2n}) \rightarrow T\overline{M}'$ be the bundle of all 2-subspaces in fibres of the induced bundle $\pi^{-1}T\overline{M}^{2n}$ of (\overline{M}^{2n}, L) . Its standard fibre is the Grassmann manifold $G_{2,2n}(\mathbb{R})$ of all 2-planes in \mathbb{R}^{2n} . Note that the vertical sectional curvature is a function $s : GF_2(\overline{M}^{2n}) \rightarrow \mathbb{R}$ rather than a function on $T\overline{M}'$. Let $(\overline{M}^{2n}, L, \overline{J})$ be a nearly Kaehlerian Finsler space; a *Finslerian 2-plane* $p \in GF_2(\overline{M}^{2n})$ is said to be *holomorphic* if $\overline{J}(p) = p$. The restriction of s to the holomorphic 2-planes is referred to as the *holomorphic \mathcal{V} -sectional curvature*. Then, $(\overline{M}^{2n}, L, \overline{J})$ is said to be a complex Finslerian \mathcal{V} -space-form [12] if there exists $c \in C^\infty(T\overline{M}')$ such that the following equality $s = c \circ \sigma$ holds on all holomorphic 2-planes $p \in GF_2(\overline{M}^{2n})$. If $(\overline{\nabla}, \overline{N})$ is a metrical Finsler connection $(\overline{M}, L, \overline{J})$ then the associated vertical curvature $\overline{R}^\mathcal{V}(X, Y, Z, W) := \overline{g}(\overline{R}^\mathcal{V}(X, Y)Z, W)$ is skew-symmetric in X, Y , respectively in Z, W , and thus the above procedure is easily generalized such as to yield a well defined concept of (holomorphic) \mathcal{V} -sectional curvature. Moreover, if the holomorphic \mathcal{V} -sectional curvature s (constructed with respect to $(\overline{\nabla}, \overline{N})$), does not depend on the 2-planes $p \in \pi_u^{-1}T\overline{M}^{2n}$ but only on the direction $u \in T\overline{M}'$, then \overline{M}^{2n} is also referred to as a complex \mathcal{V} -space form with respect to $(\overline{\nabla}, \overline{N})$. Let $(\overline{M}^{2n}, L, \overline{J})$ be a nearly Kaehlerian Finsler space-form. The vertical curvature tensor $\overline{R}^\mathcal{V}(X, Y, Z, W)$ is given by

$$\begin{aligned} \overline{R}^\mathcal{V}(X, Y, Z, W) = & \frac{c}{4} [\overline{g}(X, W)\overline{g}(Y, Z) - \overline{g}(X, Z)\overline{g}(Y, W) + \overline{g}(X, \overline{J}W)\overline{g}(Y, \overline{J}Z) \\ & - \overline{g}(X, \overline{J}Z)\overline{g}(Y, \overline{J}W) - 2\overline{g}(X, \overline{J}Y)\overline{g}(Z, \overline{J}W)] + \frac{1}{4} [\overline{g}((\overline{\nabla}_X \overline{J})W, (\overline{\nabla}_Y \overline{J})Z) \\ & - \overline{g}((\overline{\nabla}_X \overline{J})Z, (\overline{\nabla}_Y \overline{J})W) - 2\overline{g}((\overline{\nabla}_X \overline{J})Y, (\overline{\nabla}_Z \overline{J})W)], \end{aligned} \quad (2.5)$$

for all Finslerian vector fields X, Y, Z, W of \overline{M}^{2n} (see [23]). Suppose, instead that \overline{g} is non-degenerate on $T\overline{M}^{2n}$, i.e., $\text{rank}(\overline{g}) = 2n$ on any coordinate neighborhood of $T\overline{M}$. Clearly, at any point u of $T\overline{M}'$, \overline{g}_u is a pseudo-Euclidean metric on the fibre $\mathcal{V}T\overline{M}'$ at u . Denote

by q the index of \bar{g}_u , i.e., q is the dimension of the largest subspace of $\mathcal{V}T\bar{M}'$ on which \bar{g} is negative definite. We further suppose \bar{g} is of constant index, q , on $T\bar{M}$. In this case \bar{g} is said to be an *indefinite Finsler metric* and $\mathbb{F}^{2n} = (\bar{M}^{2n}, L, \bar{g})$ is called an *indefinite Finsler space* [4, 8, 9]. Furthermore, if \mathbb{F}^{2n} is of constant (holomorphic) \mathcal{V} -sectional curvature as described earlier, then we say that \mathbb{F}^{2n} is an *indefinite Finsler space-form*.

Consider a hypersurface (M, g) of \bar{M}^{2n} . From now on, we assume that \bar{g} is of index q , where $1 < q < 2n$. In this case g may be degenerate in some points of TM ; suppose g is degenerate on TM of constant rank $(2n - 1)$. Then we call M a null hypersurface of \mathbb{F}^{2n} . Consider, for each $p \in TM$, the vector space $\mathcal{V}TM_p^\perp = \{X_p \in \mathcal{V}T\bar{M}'_p : \bar{g}_p(X_p, Y_p) = 0, \forall Y_p \in \mathcal{V}TM_p\}$, and construct $\mathcal{V}TM^\perp = \cup_{p \in TM} \mathcal{V}TM_p^\perp$. Notice that $\mathcal{V}TM^\perp$ is a one dimensional vector subbundle of $\mathcal{V}T\bar{M}'|_{TM}$. Moreover, M is a null hypersurface of \mathbb{F}^{2n} if and only if $\mathcal{V}TM^\perp$ is a vector subbundle of $\mathcal{V}TM$. Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\mathcal{F}(M)$ the algebra of differentiable functions on M and by $\Gamma(E)$ the $\mathcal{F}(M)$ -module of differentiable sections of a vector bundle E over M . We also assume that all associated structures are smooth.

In the theory of non-degenerate submanifolds of a Finsler space $\mathcal{V}TM^\perp$ plays an important role in introducing main geometrical objects, such as second fundamental form, shape operator, induced connection, etc. Contrary to this situation in case of a null hypersurface, $\mathcal{V}TM^\perp$ fails to be complementary to $\mathcal{V}TM$ in $\mathcal{V}T\bar{M}'|_{TM}$. Motivated by the above, the author [4] (also see [15]) constructed a complementary (non-orthogonal) vector bundle to $\mathcal{V}TM$ in $\mathcal{V}T\bar{M}'|_{TM}$ which plays the role of $\mathcal{V}TM^\perp$. In fact, consider a complementary vector bundle $S(\mathcal{V}TM)$ of $\mathcal{V}TM^\perp$ in $\mathcal{V}TM$, i.e. we have $\mathcal{V}TM = S(\mathcal{V}TM) \perp \mathcal{V}TM^\perp$. It is easy to see that $S(\mathcal{V}TM)$ is a non-degenerate vector subbundle of $\mathcal{V}TM$, whose existence is secured by the paracompactness of M . $S(\mathcal{V}TM)$ is called the *screen distribution* [4] of M . Next, along to $S(\mathcal{V}TM)$ we have the decomposition $\mathcal{V}T\bar{M}'|_{TM} = S(\mathcal{V}TM) \perp S(\mathcal{V}TM)^\perp$, where $S(\mathcal{V}TM)^\perp$ is the complementary vector bundle to $S(\mathcal{V}TM)$ in $\mathcal{V}T\bar{M}'|_{TM}$.

Theorem 2.1 ([4]). *Let M be a null hypersurface of \mathbb{F}^{2n} and $S(\mathcal{V}TM)$ be the screen distribution of M . Then there exists a unique vector bundle $tr(\mathcal{V}TM)$ of rank 1 over TM , such that for any non-zero section ξ of $\mathcal{V}TM^\perp$ on a coordinate neighborhood $\mathcal{U} \subset TM$, there exists a unique section N of $tr(\mathcal{V}TM)$ on \mathcal{U} satisfying: $\bar{g}(N, \xi) = 1$, and $\bar{g}(N, N) = \bar{g}(N, X) = 0$, for all $X \in \Gamma(S(\mathcal{V}TM)|_{\mathcal{U}})$.*

The vector bundle $tr(\mathcal{V}TM)$ above is called the *null transversal* vector bundle of M with respect to $S(\mathcal{V}TM)$. Moreover, we have the following decomposition

$$\begin{aligned}\mathcal{V}\overline{TM}'|_{TM} &= S(\mathcal{V}TM) \perp \{\mathcal{V}TM^\perp \oplus tr(\mathcal{V}TM)\} \\ &= \mathcal{V}TM \oplus tr(\mathcal{V}TM).\end{aligned}\tag{2.6}$$

Let (M, g) be a null hypersurface of \mathbb{F}^{2n} . In view of (2.6), the author [4] proves (see Theorem 2.1) that there is a unique nonlinear connection $\mathcal{H}TM$ [4] on TM , which is a subbundle of $tr(\mathcal{V}TM) \oplus \mathcal{V}\overline{TM}^{2n}$. Accordingly, $\mathcal{H}TM$ is called the induced nonlinear connection on TM . Denote by $(\mathcal{H}TM, \nabla)$, the induced Finsler connection on M by $(\overline{N}, \overline{\nabla})$ on \overline{M}^{2n} . Locally, the Gauss and Weingarten equations of M are given by

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \quad \text{and} \quad \overline{\nabla}_X N = -A_N X + \tau(X)N,\tag{2.7}$$

for all $X \in \Gamma(TTM)$, $Y \in \Gamma(\mathcal{V}TM)$ and $N \in \Gamma(tr(\mathcal{V}TM))$. Here, B is called the local second fundamental form of M , and A_N its shape operator. furthermore, τ is a differential 1-form on TM . It then follows that ∇ is a linear connection on TM . Denote by P the projection of $\mathcal{V}TM$ onto $S(\mathcal{V}TM)$, then the local Gauss and Weingarten formulae of $S(\mathcal{V}TM)$ are given by

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi \quad \text{and} \quad \nabla_X \xi = -A_\xi^* X - \tau(X)\xi,\tag{2.8}$$

for all $X \in \Gamma(TTM)$ and $\xi \in \Gamma(\mathcal{V}TM^\perp)$. Moreover, C is the local second fundamental form of $S(\mathcal{V}TM)$, and ∇^* is a linear connection on it, which is a metric connection. In general, ∇ is not a metric connection. In fact, if η is a one form on $\mathcal{V}TM$ by $\eta(\cdot) = \overline{g}(\cdot, N)$, then ∇g is given by

$$(\nabla_X g)(Y, Z) = B(X, Z)\eta(Y) + B(X, Y)\eta(Z),\tag{2.9}$$

for all $X \in \Gamma(TTM)$ and $Y, Z \in \Gamma(\mathcal{V}TM)$. Notice that B is degenerate and in fact, $B(\cdot, E) = 0$. The shape operators A_ξ^* and A_N are screen-valued and relate to their shape local second fundamental forms according to the relations

$$B(X, Y) = g(A_\xi^* X, Y) \quad \text{and} \quad C(X, PY) = g(A_N X, PY).\tag{2.10}$$

The null hypersurface M is said to be *totally umbilic* [15] if $B = \rho \otimes g$, where ρ is a smooth function on a coordinate neighborhood $\mathcal{U} \subset TM$. In case $\rho = 0$, we say that M is *totally geodesic*. In the same line, M is called *screen totally umbilic* if $C = \varrho \otimes g$, where ϱ is a

smooth function on a coordinate neighborhood $\mathcal{U} \subset TM$. When $\varrho = 0$, we say that M is *screen totally geodesic*.

Denote by $\bar{R}^\mathcal{V}$ and R , the curvature tensors of $\bar{\nabla}$ and ∇ . Let further, ∇° be Schouten-Van Kampen connection [5] on TM and T° its torsion. Then

$$\begin{aligned} \bar{R}^\mathcal{V}(X, Y)Z &= R(X, Y)Z + A_{h(X, Z)}Y - A_{h(Y, Z)}X \\ &\quad + (\nabla_X^t h)(Y, Z) - (\nabla_Y^t h)(X, Z) - h(T^\circ(X, Y), Z), \end{aligned} \quad (2.11)$$

for any $X, Y \in \Gamma(TTM)$, and $Z \in \Gamma(\mathcal{V}TM)$. Here, $(\nabla_X^t h)(Y, Z) = \nabla_X^t h(Y, Z) - h(\nabla_X^\circ Y, Z) - h(Y, \nabla_X^\circ Z)$, where ∇^t is the transversal (linear) connection and $h(X, Y) = B(X, Y)N$. Further details regarding the fundamental equations of null hypersurfaces can be found in [4, 15, 16].

As seen above, let ξ and N the metric normal and the transversal sections, respectively. Since (\bar{g}, \bar{J}) is an almost Hermitian structure and $\bar{J}\xi$ is a null vector field, it follows that $\bar{J}N$ is null too. Moreover, $\bar{g}(\bar{J}\xi, \xi) = 0$ and, thus, $\bar{J}\xi$ is tangent to TM . Let us consider $S(\mathcal{V}TM)$ containing $\bar{J}\mathcal{V}TM^\perp$ as a vector subbundle. Consequently, N is orthogonal to $\bar{J}\xi$ and we have $\bar{g}(\bar{J}N, \xi) = -\bar{g}(N, \bar{J}\xi) = 0$ and $\bar{g}(\bar{J}N, N) = 0$. This means that $\bar{J}N$ is tangent to TM and in particular, it belongs to $S(\mathcal{V}TM)$. Thus, $\bar{J}tr(\mathcal{V}TM)$ is also a vector subbundle of $S(\mathcal{V}TM)$. In view of (2.2), we have $\bar{g}(\bar{J}\xi, \bar{J}N) = 1$. It is then easy to see that $\bar{J}\mathcal{V}TM^\perp \oplus \bar{J}tr(\mathcal{V}TM)$ is a non-degenerate vector subbundle of $S(\mathcal{V}TM)$, with 2-dimensional fibers. Then there exists a non-degenerate distribution D_0 on TM such that $S(\mathcal{V}TM) = \{\bar{J}\mathcal{V}TM^\perp \oplus \bar{J}tr(\mathcal{V}TM)\} \perp D_0$. It is easy to check that D_0 is an almost complex distribution with respect to \bar{J} , i.e. $\bar{J}D_0 = D_0$. The decomposition of $\mathcal{V}TM$ becomes $\mathcal{V}TM = \{\bar{J}\mathcal{V}TM^\perp \oplus \bar{J}tr(\mathcal{V}TM)\} \perp D_0 \perp \mathcal{V}TM^\perp$. If we set $D := \mathcal{V}TM^\perp \perp \bar{J}\mathcal{V}TM^\perp \perp D_0$ and $D' = \bar{J}tr(\mathcal{V}TM)$, then $\mathcal{V}TM = D \oplus D'$. Here, D is an almost complex distribution and D' is carried by \bar{J} just into the transversal bundle. Thus, we have a null CR submanifold as in [15, 16] for null hypersurfaces of semi-Riemannian manifolds. Finally, let us set

$$U := -\bar{J}N \quad \text{and} \quad V := -\bar{J}\xi. \quad (2.12)$$

3. TOTALLY UMBILIC HYPERSURFACES

In this section, we prove several characterization results on umbilic hypersurfaces of an indefinite nearly Kaehlerian Finsler space-form $\mathbb{F}^{2n} := (\bar{M}(c)^{2n}, L, \bar{g}, \bar{J})$. To that end, we have the following.

Theorem 3.1. *Let \mathbb{F}^{2n} be an indefinite $2n$ -dimensional nearly Kaehlerian Finsler space-form, admitting a totally umbilic null hypersurface (M, g) . Then, c satisfies $c > 0$, $c = 0$ or $c < 0$ if and only if the vector field $\nabla_U^* V$ is timelike, null (or identically zero), or spacelike, respectively. Moreover, the umbilicity factor ρ satisfies the differential equations $\xi\rho + \rho\tau(\xi) - \rho^2 = 0$ and $PX\rho + \rho\tau(PX) = 0$. for any $X \in \Gamma(\mathcal{V}TM)$.*

Proof. Setting $Y = W = \xi$ and $X = Z$ in (2.5) and using (2.12), we derive

$$\bar{R}^\mathcal{V}(Z, \xi, Z, \xi) = -\frac{3c}{4}g(Z, V)^2 - \frac{3}{4}\bar{g}((\bar{\nabla}_Z \bar{J})\xi, (\bar{\nabla}_Z \bar{J})\xi). \quad (3.13)$$

A straightforward calculation, while considering (2.7), (2.8) and (2.12), leads to

$$(\bar{\nabla}_Z \bar{J})\xi = -\nabla_Z^* V - C(Z, V)\xi - \rho g(Z, V)N + \rho \bar{J}Z - \tau(Z)V. \quad (3.14)$$

In view of (3.14), the second term on the right hand side of (3.13) becomes

$$\begin{aligned} -\frac{3}{4}\bar{g}((\bar{\nabla}_Z \bar{J})\xi, (\bar{\nabla}_Z \bar{J})\xi) &= -\frac{3}{4}g(\nabla_Z^* V, \nabla_Z^* V) + \frac{3}{2}\rho g(\nabla_Z^* V, \bar{J}Z) \\ &\quad + \frac{3\rho^2}{2}g(Z, U)g(Z, V) - \frac{3\rho^2}{4}g(Z, Z). \end{aligned} \quad (3.15)$$

On the other hand, setting $Y = \xi$ and $X = Z$ in (2.11) and taking the inner product of the resulting equation with ξ , we get

$$\begin{aligned} \bar{R}^\mathcal{V}(Z, \xi, Z, \xi) &= \bar{g}((\nabla_Z^t h)(\xi, Z) - (\nabla_\xi^t h)(Z, Z) - h(T^\circ(Z, \xi), Z), \xi) \\ &= (\rho^2 - \rho\tau(\xi) - \xi\rho)g(Z, Z). \end{aligned} \quad (3.16)$$

Using (3.13), (3.15) and (3.16), we get

$$\begin{aligned} (\rho^2 - \rho\tau(\xi) - \xi\rho)g(Z, Z) &= -\frac{3c}{4}g(Z, V)^2 - \frac{3}{4}g(\nabla_Z^* V, \nabla_Z^* V) \\ &\quad + \frac{3}{2}\rho g(\nabla_Z^* V, \bar{J}Z) + \frac{3\rho^2}{2}g(Z, U)g(Z, V) - \frac{3\rho^2}{4}g(Z, Z), \end{aligned} \quad (3.17)$$

in which we have used the fact that the adapted connection ∇° coincides with ∇ . Setting $Z = U$ and $Z = U + V$ in (3.17) in turn, we get

$$c + g(\nabla_U^* V, \nabla_U^* V) = 0, \quad (3.18)$$

$$\text{and } 2(\xi\rho + \rho\tau(\xi) - \rho^2) = -\frac{3c}{4} - \frac{3}{4}g(\nabla_{U+V}^* V, \nabla_{U+V}^* V). \quad (3.19)$$

By considering the facts $(\bar{\nabla}_V \bar{J})V = 0$ and that M is totally umbilic, we derive $\nabla_V^* V = [\rho - C(V, V)]\xi - \tau(V)V$. Thus, $g(\nabla_{U+V}^* V, \nabla_{U+V}^* V) = g(\nabla_U^* V, \nabla_U^* V)$ and (3.18) and (3.19) implies that $\xi\rho + \rho\tau(\xi) - \rho^2 = 0$, proving the first assertions of the theorem and the first differential differential equation of ρ .

Next, we prove the second differential equation of the theorem. To that end, (2.5) and (2.11) implies

$$\begin{aligned} [X\rho + \rho\tau(X)]g(Y, Z) - [Y\rho + \rho\tau(Y)]g(X, Z) &= \frac{c}{4}[-\bar{g}(X, V)\bar{g}(Y, \bar{J}Z) \\ &+ \bar{g}(Y, V)\bar{g}(X, \bar{J}Z) + 2\bar{g}(Z, V)\bar{g}(X, \bar{J}Y)] + \frac{1}{4}[\bar{g}((\bar{\nabla}_X \bar{J})\xi, (\bar{\nabla}_Y \bar{J})Z) \\ &- \bar{g}((\bar{\nabla}_Y \bar{J})\xi, (\bar{\nabla}_X \bar{J})Z) - 2\bar{g}((\bar{\nabla}_Z \bar{J})\xi, (\bar{\nabla}_X \bar{J})Y)], \end{aligned} \quad (3.20)$$

for all $X, Y, Z \in \Gamma(S(\mathcal{V}TM))$. Setting $Y = Z = V$ in (3.20), we get

$$-[V\rho + \rho\tau(V)]\bar{g}(X, V) = -\frac{3}{4}\bar{g}((\bar{\nabla}_V \bar{J})\xi, (\bar{\nabla}_X \bar{J})V). \quad (3.21)$$

Using (2.4), we see that $(\bar{\nabla}_V \bar{J})V = 0$, which helps to derive

$$-\bar{\nabla}_V V = -\rho\xi + \tau(V)V. \quad (3.22)$$

Thus, in view of (2.7), (2.8) and (2.12), we have

$$(\bar{\nabla}_V \bar{J})\xi = -\bar{\nabla}_V V + \bar{J}A_\xi^*V - \tau(V)V = -\bar{\nabla}_V V + \rho\xi - \tau(V)V. \quad (3.23)$$

Hence, considering (3.21), (3.22) and (3.23), we get

$$-[V\rho + \rho\tau(V)]\bar{g}(X, V) = 0, \quad \forall X \in \Gamma(S(\mathcal{V}TM)). \quad (3.24)$$

Setting $X = U$ in (3.24), we get

$$V\rho + \rho\tau(V) = 0. \quad (3.25)$$

On the other hand, setting $X = V$ and $Y = Z = U$ in (3.20), we derive

$$-[U\rho + \rho\tau(U)] = \frac{3}{4}\bar{g}((\bar{\nabla}_U \bar{J})\xi, (\bar{\nabla}_U \bar{J})V). \quad (3.26)$$

A straightforward calculation, using (2.7), (2.8) and (2.12), we derive $(\bar{\nabla}_U \bar{J})\xi = -\nabla_U V - \tau(U)V$ and $(\bar{\nabla}_U \bar{J})V = -\bar{J}\nabla_U V - \tau(U)\xi$. Thus, (3.26) gives

$$U\rho + \rho\tau(U) = 0. \quad (3.27)$$

Next, let $Y = Z = \bar{J}X$ in (3.20), for some $X \in \Gamma(D_0)$, we get

$$[X\rho + \rho\tau(X)]g(\bar{J}X, \bar{J}X) = -\frac{3}{4}\bar{g}((\bar{\nabla}_X \bar{J})\bar{J}X, (\bar{\nabla}_{\bar{J}X} \bar{J})\xi). \quad (3.28)$$

As $0 = (\bar{\nabla}_X \bar{J})X = \bar{\nabla}_X \bar{J}X - \bar{J}\bar{\nabla}_X X$ by (2.4), we have $\bar{J}\bar{\nabla}_X \bar{J}X + \bar{\nabla}_X X = 0$. Using this relation, we derive

$$(\bar{\nabla}_X \bar{J})\bar{J}X = \bar{\nabla}_X \bar{J}^2 X - \bar{J}\bar{\nabla}_X \bar{J}X = -\bar{\nabla}_X X - \bar{J}\bar{\nabla}_X \bar{J}X = 0. \quad (3.29)$$

Considering (3.28) and (3.29), we get

$$X\rho + \rho\tau(X) = 0, \quad \forall X \in \Gamma(D_0). \quad (3.30)$$

Hence, part (3) of the theorem follows from (3.25), (3.27) and (3.30), which completes the proof.

As consequence, we have the following results.

Corollary 3.1. *Let \mathbb{F}^{2n} be an indefinite nearly Kaehlerian Finsler space-form, such that $\bar{J}\bar{\nabla} = 0$. If \mathbb{F}^{2n} admits a totally umbilic null hypersurface (M, g) , then $c = 0$.*

Proof. From (3.18), we have $c + g(\nabla_U^* V, \nabla_U^* V) = 0$. Also, from the assumption $\bar{J}\bar{\nabla} = 0$, we see that $(\bar{\nabla}_U \bar{J})V = 0$, which implies that $-\bar{J}\nabla_U^* V = \tau(U)\xi - C(U, V)V$, in which we have used (2.7) and (2.8). Thus, $g(\nabla_U^* V, \nabla_U^* V) = \bar{g}(\bar{J}\nabla_U^* V, \bar{J}\nabla_U^* V) = 0$, which gives $c = 0$.

Corollary 3.2. *Let \mathbb{F}^{2n} be an indefinite $2n$ -dimensional nearly Kaehlerian Finsler space-form. If \mathbb{F}^{2n} admits a totally umbilic null hypersurface (M, g) , such that V is parallel with respect to ∇^* , then $c = 0$.*

Proof. From (3.18), we have $c = 0$. By (3.17), we have $2\rho^2 g(Z, U)g(Z, V) - \rho^2 g(Z, Z) = 0$, for all $Z \in \Gamma(\mathcal{V}TM)$. Setting $Z = X \in \Gamma(D_0)$ in this relation and noticing that $g(X, U) = g(X, V) = 0$, we get $\rho = 0$. Thus, M is totally geodesic which completes the proof.

Corollary 3.3. *Let \mathbb{F}^{2n} be an indefinite nearly Kaehlerian Finsler space-form, admitting a totally umbilic null hypersurface (M, g) . If M is also screen totally umbilic, then M is screen totally geodesic.*

Proof. By a straightforward calculation, we have $g(\nabla_U \xi, U) = \bar{g}(\bar{\nabla}_U \xi, U) = \bar{g}(\bar{J}\bar{\nabla}_U \xi, N)$. In view of (2.4), we have $\bar{J}\bar{\nabla}_U \xi = \bar{\nabla}_U \bar{J}\xi + (\bar{\nabla}_\xi \bar{J})U$, and the previous relation simplifies to

$$\begin{aligned} g(\nabla_U \xi, U) &= \bar{g}(\bar{\nabla}_U \bar{J}\xi, N) + \bar{g}((\bar{\nabla}_\xi \bar{J})U, N) \\ &= -\bar{g}(\bar{J}\xi, \bar{\nabla}_U N) + \bar{g}(\bar{\nabla}_\xi N, N) - \bar{g}(\bar{\nabla}_\xi U, U) = \varrho. \end{aligned} \quad (3.31)$$

But using (2.8), we see that $g(\nabla_U \xi, U) = -B(U, U) = -\rho g(U, U) = 0$. Thus, in view of (3.31), we get $\varrho = 0$, showing that M is screen totally geodesic which completes the proof.

Let $x \in M$ and ξ be a null vector of $T_x M$. A plane H of $T_x M$ is called a null plane directed by ξ if it contains ξ , $g_x(\xi, W) = 0$ for any $W \in H$ and there exists $W_0 \in H$ such that $g_x(W_0, W_0) \neq 0$. Thus, the null section curvature of H with respect to ξ and the induced connection ∇ of M , is defined as a real number $K_\xi(H) = g_x(R(W, \xi)\xi, W)/g_x(W, W)$, where $W \neq 0$ is any vector in H independent with ξ (see [15] or [16] for more details). Moreover, the author in [20] proved that an n (where $n \geq 3$)-dimensional Lorentzian manifold is of constant curvature if and only if its null sectional curvatures are everywhere zero.

Theorem 3.2. *Let \mathbb{F}^{2n} be an indefinite $2n$ -dimensional nearly Kaehlerian Finsler space-form, admitting a totally umbilic and screen totally umbilic null hypersurface (M, g) . Then, the null sectional curvature $K_\xi(H)$ of M vanishes if and only if V is parallel with respect to ∇^* .*

Proof. Considering (2.11) and Corollary 3.3, we have

$$g(R(X, Y)Z, PW) = \bar{g}(\bar{R}(X, Y)Z, PW), \quad (3.32)$$

for all $X, Y \in \Gamma(TTM)$ and $Z, W \in \Gamma(\mathcal{V}TM)$. Setting $X = PW = W$ and $Y = Y = \xi$ in (3.32) and using (2.5), we have

$$g(R(W, \xi)\xi, W) = \frac{3c}{4}g(W, V)^2 + \frac{3}{4}\bar{g}((\bar{\nabla}_W \bar{J})\xi, (\bar{\nabla}_W \bar{J})\xi). \quad (3.33)$$

But in view of (2.8) and (2.12), we have $(\bar{\nabla}_W \bar{J})\xi = -\bar{\nabla}_W V + \rho \bar{J}W - \tau(W)V$. Thus, from (3.33), we have

$$\begin{aligned} \bar{g}((\bar{\nabla}_W \bar{J})\xi, (\bar{\nabla}_W \bar{J})\xi) &= \bar{g}(\bar{\nabla}_W V, \bar{\nabla}_W V) - 2\rho \bar{g}(\bar{\nabla}_W V, \bar{J}W) + \rho^2 g(W, W) \\ &= g(\nabla_W^* V, \nabla_W^* V) - 2\rho g(\nabla_W^* V, \bar{J}W) \\ &\quad - 2\rho^2 g(W, U)g(W, V) + \rho^2 g(W, W). \end{aligned} \quad (3.34)$$

Suppose that V is parallel with respect to ∇^* , then the term $g(\nabla_W^* V, \nabla_W^* V)$ vanishes. Moreover, $-2\rho g(\nabla_W^* V, \bar{J}W)$ vanishes too. Furthermore, by Corollary 3.2, we have $c = 0$ and $\rho = 0$. Then, (3.34) reduces to

$$\bar{g}((\bar{\nabla}_W \bar{J})\xi, (\bar{\nabla}_W \bar{J})\xi) = -2\rho^2 g(W, U)g(W, V) + \rho^2 g(W, W) = 0. \quad (3.35)$$

Hence, considering (3.35) in (3.33), we see that $K_\xi(H) = 0$. The converse is obvious, and the proof is complete.

The following result follows easily from Theorem 3.2.

Corollary 3.4. *Let \mathbb{F}^{2n} be an indefinite $2n$ -dimensional normal nearly Kaehlerian Finsler space-form, admitting a totally umbilic null hypersurface (M, g) . Then, the null sectional curvature $K_\xi(H)$ of M vanishes.*

4. GEOMETRY OF (M, g) FROM THE DISTRIBUTIONS D AND D'

In this section, we give new results on the null hypersurface (M, g) based on the nature of D and D' with respect to the second fundamental form B of M . Denote by Q the projection morphism of TM onto D . Then, in view of (2.12), any $X \in \Gamma(\mathcal{V}TM)$ can be written as $X = QX + u(X)U$, where u is a 1-form locally defined on M by $u(\cdot) = g(\cdot, V)$. Applying \bar{J} to this relation we have

$$\bar{J}X = FX + u(X)N, \quad (4.36)$$

where F is a $(1,1)$ -tensor globally defined on M by $F = \bar{J} \circ Q$. Moreover, it is easy to see that (F, u, U) is a local almost contact structure on M , satisfying

$$F^2 = -I + u \otimes U, \quad u(U) = 1. \quad (4.37)$$

Notice that (F, u, U) is never an almost contact metric structure on with respect to degenerate metric g . Using (2.4), (2.7) and (2.8), we derive

$$(\nabla_X F)Y + (\nabla_Y F)X = u(Y)A_N X + u(X)A_N Y - 2B(X, Y)U, \quad (4.38)$$

and

$$\begin{aligned} B(X, FY) + B(Y, FX) &= -B(X, V)\eta(Y) - B(Y, V)\eta(X) \\ &\quad - g(\nabla_X V, Y) - g(\nabla_Y V, X) - u(X)\tau(Y) - u(Y)\tau(X), \end{aligned} \quad (4.39)$$

for any $X, Y \in \Gamma(\mathcal{V}TM)$.

Theorem 4.1. *Let \mathbb{F}^{2n} be an indefinite $2n$ -dimensional normal nearly Kaehlerian Finsler space-form, admitting a totally umbilic and screen totally umbilic null hypersurface (M, g) . If F is parallel then $c = 0$.*

Proof. Suppose that F is parallel. Then, (4.38) and the facts M is totally umbilic and screen totally umbilic, gives $\rho u(Y)X + \rho u(X)Y = 2\rho g(X, Y)U$. In view of Corollary 3.3, M is screen totally geodesic and thus, we have $2\rho g(X, Y)U = 0$. Hence, $\rho = 0$ and M is totally geodesic. On the other hand, $(\nabla_X F)\xi = 0$ and (2.8) gives $\nabla_X V = -\tau(X)V$, which together with (3.18) gives $c = 0$. This completes the proof.

In [6], the authors introduced the concept of *mixed geodesic* non-degenerate CR-submanifolds of a space form. More precisely, a CR-submanifold is called mixed geodesic if its second fundamental h satisfies $h(X, Y) = 0$, where $X \in \Gamma(D)$ and $Y \in \Gamma(D^\perp)$. Here, D is a \bar{J} -invariant distribution on M and D^\perp is an anti-invariant distribution which is orthogonal and complementary to D in M . As we have already secured a CR structure on a null hypersurface (M, g) (see Section 2), in which the invariant distribution D is given by $D = \mathcal{V}TM^\perp \perp \bar{J}\mathcal{V}TM^\perp \perp D_0$ and its complementary (but not orthogonal) distribution D' by $D' = \bar{J}tr(\mathcal{V}TM)$, we can define the concept of mixed geodesic for M as follows.

Definition 4.1. Let (M, g) be a null hypersurface of a complex space. Then, M is said to be mixed totally geodesic if $B(X, Y) = 0$, for $X \in \Gamma(D)$ and $Y \in \Gamma(D')$.

It follows that any totally geodesic null hypersurface (M, g) of \mathbb{F}^{2n} is trivially mixed geodesic.

Notice that the distribution D' is integrable while D is generally non-integrable. In fact, it is easy to show, using (2.4) and (2.7), that D is integrable if and only if

$$B(X, \bar{J}Y) - B(Y, \bar{J}X) = 2\bar{g}((\bar{\nabla}_X \bar{J})Y, \xi), \quad (4.40)$$

for all $X, Y \in \Gamma(D)$. A null hypersurface (M, g) will be called *mixed foliate* if M is mixed totally geodesic and (4.40) holds, i.e. D is integrable. Since D and D' are not g -orthogonal distributions, one may not be able to describe the nature of (M, g) depending on the leaves M_D and $M_{D'}$ of D , assumed integrable, and D' , respectively. However, when M is mixed foliate, we know that $D \perp_B D'$. This prompts us to consider the null hypersurface (M, B) , that is; the null hypersurface M endowed with its local second fundamental form B , instead of its natural degenerate metric g . Notice that (M, B) is also degenerate since the second fundamental form B is degenerate. More precisely, $B(\xi, \cdot) = 0$. It is easy to see that the radical distribution of (M, B) is $\ker A_\xi^*$. Such hypersurfaces were also studied by C. L. Bejan and K. L. Duggal [7]. A distribution \mathcal{D} on M will be called B -totally null if B vanishes on \mathcal{D} . It follows that a totally geodesic M is B -totally null hypersurface.

The following lemma is fundamental to our study of (M, g) and (M, B) .

Lemma 4.1. Let (M, g) be a mixed foliate null hypersurface of \mathbb{F}^{2n} . Then,

$$\begin{aligned} 2cg(Y, Y) &= 4B(\bar{J}Y, \nabla_Y U) - 4B(Y, \nabla_{\bar{J}Y} U) \\ &\quad + \bar{g}((\bar{\nabla}_{\bar{J}Y} \bar{J})U, (\bar{\nabla}_Y \bar{J})\xi) - \bar{g}((\bar{\nabla}_Y \bar{J})U, (\bar{\nabla}_{\bar{J}Y} \bar{J})\xi), \end{aligned}$$

for all $Y \in \Gamma(D_0)$.

Proof. For any $X, Y \in \Gamma(D)$, we have

$$(\nabla_X B)(Y, U) = -B(\nabla_X Y, U) - B(Y, \nabla_X U), \quad (4.41)$$

where we have used the fact that M is mixed totally geodesic. Then, interchanging X and Y in (4.41) and subtracting the new relation from (4.41), we get

$$\begin{aligned} (\nabla_X B)(Y, U) - (\nabla_Y B)(X, U) \\ = -B([X, Y], U) + B(X, \nabla_Y U) - B(Y, \nabla_X U) \\ = B(X, \nabla_Y U) - B(Y, \nabla_X U). \end{aligned} \quad (4.42)$$

In view of (4.42) and (2.11), we derive

$$\bar{g}(\bar{R}^\mathcal{V}(X, Y)U, \xi) = B(X, \nabla_Y U) - B(Y, \nabla_X U), \quad (4.43)$$

for any $X, Y \in \Gamma(D)$. On the other hand, (2.5) gives

$$\begin{aligned} \bar{R}^\mathcal{V}(X, Y, U, \xi) &= \frac{c}{2}g(X, \bar{J}Y) + \frac{1}{4}[\bar{g}((\bar{\nabla}_X \bar{J})\xi, (\bar{\nabla}_Y \bar{J})U) \\ &\quad - \bar{g}((\bar{\nabla}_X \bar{J})U, (\bar{\nabla}_Y \bar{J})\xi) - 2\bar{g}((\bar{\nabla}_X \bar{J})Y, (\bar{\nabla}_U \bar{J})\xi)]. \end{aligned} \quad (4.44)$$

Notice that $(\bar{\nabla}_{\bar{J}Y} \bar{J})Y = 0$, for any $Y \in \Gamma(D_0)$. In fact, by (2.4) we have $(\bar{\nabla}_Y \bar{J})Y = 0$, which implies that $\bar{\nabla}_Y \bar{J}Y = \bar{J} \bar{\nabla}_Y Y$. Then, using this relation, we derive

$$(\bar{\nabla}_{\bar{J}Y} \bar{J})Y = -(\bar{\nabla}_Y \bar{J})\bar{J}Y = \bar{\nabla}_Y Y + \bar{J}^2 \bar{\nabla}_Y Y = 0. \quad (4.45)$$

Setting $X = \bar{J}Y$, where $Y \in \Gamma(D_0)$, in (4.43) and (4.44), and considering (4.45), we get the lemma. Hence, the proof.

As a consequence of Lemma 4.1, we have the following result.

Theorem 4.2. *Let (M, g) be a mixed foliate null hypersurface of an indefinite nearly Kaehlerian Finsler space-form \mathbb{F}^{2n} , such that F is parallel. Then $c = 0$. Moreover, (M, B) is locally a null product manifold $M_D \times M_{D'}$, where M_D is a leaf of the invariant distribution D , which is a B -totally null manifold and $M_{D'}$ is a curve of the anti-invariant D' .*

Proof. The assumption F is parallel implies that

$$\nabla_X U = u(\nabla_X U)U, \quad \forall X \in \Gamma(\mathcal{V}TM). \quad (4.46)$$

Using (4.46) and the fact that M is mixed foliate, we derive

$$B(\bar{J}Y, \nabla_Y U) = u(\nabla_Y U)B(\bar{J}Y, U) = 0, \quad (4.47)$$

$$\text{and } B(Y, \nabla_{\bar{J}Y} U) = u(\nabla_{\bar{J}Y} U)B(Y, U) = 0, \quad (4.48)$$

for any $Y \in \Gamma(D_0)$. On the other hand, applying (2.7), (2.8) and the assumption F is parallel, we have

$$\begin{aligned} (\bar{\nabla}_X \bar{J})Y &= -u(Y)A_N X + B(X, Y)U + [B(X, FY) \\ &\quad + Xu(Y) + u(Y)\tau(X) - u(\nabla_X Y)]N, \end{aligned} \quad (4.49)$$

for any $X, Y \in \Gamma(\mathcal{V}TM)$. Replacing Y by U and ξ in (4.49), in turns, we get

$$(\bar{\nabla}_X \bar{J})U = -A_N X + [\tau(X) - u(\nabla_X U)]N, \quad (4.50)$$

$$\text{and } (\bar{\nabla}_X \bar{J})\xi = -[B(X, V) + u(\nabla_X \xi)]N = 0, \quad (4.51)$$

respectively, for any $X \in \Gamma(\mathcal{V}TM)$. Considering (4.47), (4.48), (4.50) and (4.51) in Lemma 4.1, we get $2cg(Y, Y) = 0$ for any $Y \in \Gamma(D_0)$, which implies that $c = 0$ as D_0 is non-degenerate. Next, let us consider the manifold (M, B) . As F is parallel, (4.38) implies that

$$2B(X, Y)U = u(Y)A_N X + u(X)A_N Y, \quad (4.52)$$

for any $X, Y \in \Gamma(\mathcal{V}TM)$. Since u vanishes on D , (4.52) implies $B(X, Y) = 0$, for any $X, Y \in \Gamma(D)$. Thus, the distribution D is totally B -degenerate. Furthermore, using (4.52), we derive $B(U, U) = C(U, V)$ and $A_N V = 0$. From (4.46), D' is parallel. Notice that FX has no component in D' for any $X \in \Gamma(\mathcal{V}TM)$. In fact, by (4.36) and (2.12), we have $g(FX, V) = \bar{g}(\bar{J}X, V) = -\bar{g}(\bar{J}X, \bar{J}\xi) = -g(X, \xi) = 0$, i.e. $FX \in \Gamma(D)$ for all $X \in \Gamma(\mathcal{V}TM)$. Using this fact and the assumption that F is parallel, we have $\nabla_X FY = F\nabla_X Y \in \Gamma(D)$, for any $X \in \Gamma(\mathcal{V}TM)$ and $Y \in \Gamma(D)$, hence, D is parallel too. Consequently, since $D \perp_B D'$ by the mixed geodesic assumption and that D and D' are integrable distributions, then by the arguments originally used by de Rham [10], (M, B) is locally a semi-Riemannian product $M_D \times M_{D'}$, where M_D is a leaf of the invariant distribution D , which is a totally null manifold with respect to B , and $M_{D'}$ is a curve of the anti-invariant distribution D' , which completes the proof.

The following results follows immediately from Theorem 4.2.

Corollary 4.1. *There exist no mixed foliate null hypersurface (M, g) of an indefinite nearly Kaehlerian Finsler space-form \mathbb{F}^{2n} , such that F is parallel and $c \neq 0$.*

Corollary 4.2. *Let (M, g) be a mixed foliate null hypersurface of an indefinite nearly Kaehlerian Finsler space-form \mathbb{F}^{2n} , such that F is parallel and $C(U, V) = 0$. Then, the following holds;*

- (1) $c = 0$,
- (2) (M, B) is locally a null product manifold $M_D \times M_{D'}$, where M_D is a leaf of the invariant distribution D , which is a B -totally null manifold, and $M_{D'}$ is a B -null curve of the anti-invariant D' ,
- (3) each leaf M_D carries a Kaehlerian structure $(F, g|_D)$.

In case \bar{J} is parallel, such that both D' and $\bar{J}(\mathcal{V}TM^\perp)$ are parallel distributions on M , we have the following result.

Theorem 4.3. *Let (M, g) be a mixed foliate null hypersurface of an indefinite nearly Kaehlerian Finsler space-form \mathbb{F}^{2n} , such that $\bar{\nabla}\bar{J} = 0$ and the distribution D' is parallel. Then, $c = 0$. Moreover, if $\bar{J}(\mathcal{V}TM^\perp)$ is also parallel, then the following holds;*

- (1) if $\tau = 0$, the type numbers of M and $S(\mathcal{V}TM)$ satisfies $t_M(x) \leq 1$ and $t_{S(\mathcal{V}TM)}(x) \leq 1$, for any $x \in M$,
- (2) (M, B) is locally a null product $M_D \times M_{D'}$, where M_D is a leaf of D , which is a B -totally degenerate manifold of complex dimension, and $M_{D'}$ is a B -null curve of D' ,
- (3) (M, g) is totally geodesic,
- (4) $(F, g|_D)$ is a Kaehlerian structure on the leaf M_D .

Proof. As $\bar{\nabla}\bar{J} = 0$, we have

$$(\nabla_X F)Y = u(Y)A_N X - B(X, Y)U, \quad (4.53)$$

$$\text{and } (\nabla_X u)Y = -B(X, FY) - u(Y)\tau(X), \quad \forall X, Y \in \Gamma(\mathcal{V}TM), \quad (4.54)$$

in which we have used (2.7), (2.8) and (4.36). Setting $Y = U$ in (4.53) and (4.54), we, respectively, get

$$\nabla_X U = FA_N X + \tau(X)U \quad \text{and} \quad \tau(X) = u(\nabla_X U). \quad (4.55)$$

On the other hand, setting $Y = \xi$ in (4.53) and taking the second relation of (4.55), we get

$$\nabla_X V = FA_\xi^* X - \tau(X)V. \quad (4.56)$$

First, suppose that D' is parallel, then by (4.55) we have

$$FA_N X = 0 \quad \text{and} \quad \nabla_X U = \tau(X)U, \quad \forall X \in \Gamma(\mathcal{V}TM). \quad (4.57)$$

Considering (4.57) in Lemma 4.1, we get

$$cg(Y, Y) = 2\tau(Y)B(\bar{J}Y, U) - 2\tau(\bar{J}Y)B(Y, U) = 0, \quad (4.58)$$

for all $Y \in \Gamma(D_0)$, in which we have used the fact that M is mixed geodesic. As D_0 is non-degenerate, (4.58) gives $c = 0$.

Now, suppose that $\bar{J}(\mathcal{V}TM^\perp)$ is parallel, then (4.56) implies $FA_\xi^* X = 0$ and $\nabla_X V = -\tau(X)V$, for any $X \in \Gamma(\mathcal{V}TM)$. Equivalently, we have

$$A_\xi^* X = B(X, V)U \quad \text{and} \quad \nabla_X V = -\tau(X)V. \quad (4.59)$$

Notice that, as D' and $\bar{J}(\mathcal{V}TM^\perp)$ are parallel distributions on M , we see, from (4.57) and (4.59), that the vector fields U and V are parallel with respect to ∇ if and only if $\tau = 0$. Hence, part (1) follows easily from Corollary 1 of [14, p. 184]. Furthermore, it is easy to show that D is also parallel. In fact, as D is integrable, (4.40) and $\bar{\nabla}\bar{J} = 0$ implies that $B(X, FY) = B(Y, FX)$, for any $X, Y \in \Gamma(D)$. Setting $Y = \xi$ in this relation and using the facts $-V = F\xi$ and $B(\xi, FX) = 0$, we get $B(X, V) = 0$, for any $X \in \Gamma(D)$. Thus, the first relation of (4.59) gives

$$A_\xi^* X = 0, \quad \forall X \in \Gamma(D). \quad (4.60)$$

In view of (4.60) and (4.53), we have

$$(\nabla_X F)Y = 0, \quad \forall X, Y \in \Gamma(D). \quad (4.61)$$

Using (4.61) together with the fact $FX \in \Gamma(D)$ for any $X \in \Gamma(\mathcal{V}TM)$, we get $\nabla_X FY = F\nabla_X Y \in \Gamma(D)$. Hence, D is parallel. Since $D \perp_B D'$ by the assumption of mixed geodesic, and that D and D' are integrable, we see that (M, B) is locally a product manifold $M_D \times M_{D'}$, in which we have considered de Rham's [10] arguments on the existence of product manifolds. Here, M_D is a B -totally null leaf of D as $B = 0$ on D (see (4.60)), and $M_{D'}$ is a B -null curve of D' , as $B = 0$ on D' by (4.59) and the fact M is mixed geodesic. Furthermore, M_D has complex dimension since D is an invariant distribution. We have also seen that $B = 0$ on

both D and D' , which means that M is totally geodesic, proving part (3). Finally, part (4) follows from (4.37) and (4.61), which completes the proof.

Corollary 4.3. *The only mixed foliate null hypersurfaces of an indefinite nearly Kaehlerian Finsler space-form \mathbb{F}^{2n} , such that $\bar{\nabla}\bar{J} = 0$ and the distributions D' and $\bar{J}(\mathcal{V}TM)$ parallel are the geodesic ones.*

By definition of Lie derivative along vector fields, we have

$$(\mathcal{L}_V g)(X, Y) = B(V, Y)\eta(X) + B(V, X)\eta(Y) + g(\nabla_X V, Y) + g(X, \nabla_Y V), \quad \forall X, Y \in \Gamma(\mathcal{V}TM), \quad (4.62)$$

in which we have used (2.9). In view of (4.39), we can rewrite (4.62) as

$$(\mathcal{L}_V g)(X, Y) = -B(X, FY) - B(Y, FX) - u(X)\tau(Y) - u(Y)\tau(X). \quad (4.63)$$

In case the distribution $\bar{J}(\mathcal{V}TM^\perp)$ is killing, (4.63) becomes

$$B(X, FY) + B(Y, FX) + u(X)\tau(Y) + u(Y)\tau(X) = 0. \quad (4.64)$$

In particular, if $X, Y \in \Gamma(D)$, (4.64) gives $B(X, FY) + B(Y, FX) = 0$. Moreover, if $\bar{J}\bar{\nabla} = 0$ and D is integrable, then the last relation and (4.40) and (4.37) gives $B(X, Y) = 0$, for all $X, Y \in \Gamma(D)$. Furthermore, if D' is parallel, we see from (4.53) that $B(X, U) = C(X, V)$, for all $X \in \Gamma(\mathcal{V}TM)$. Also note, from (4.53) and the facts $B = 0$ on D and $FX \in \Gamma(D)$ for any $X \in \Gamma(\mathcal{V}TM)$, that the distribution D is parallel. Thus, putting all the above together, and considering Theorem 4.3, we have the following result.

Corollary 4.4. *Under the assumptions of Theorem 4.3, if instead the distribution $\bar{J}(\mathcal{V}TM^\perp)$ is killing then (M, B) is locally a null product $M_D \times M_{D'}$, where M_D is a leaf of D , which is a B -totally degenerate manifold of complex dimension, and $M_{D'}$ is a B -non-null curve of D' , unless $C(U, V) = 0$.*

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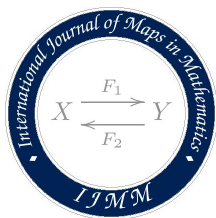
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DECOMPOSITION ON *QTAG*-MODULES WITH CERTAIN SUBMODULES

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ABSTRACT. A module M over an associative ring R with unity is a *QTAG*-module if every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules. In this paper, we study some decomposition results on *QTAG*-modules with certain submodules in terms of the cardinality $g(M)$.

1. INTRODUCTION AND TERMINOLOGY

Let R be any ring with unity. A uniserial module M is a module over a ring R , whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M , either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R -module M with a unique decomposition series, $d(M)$ denotes its decomposition length.

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A module M_R is called a *TAG*-module if it satisfies the following two conditions:

- (I) Every finitely generated submodule of every homomorphic image of M is a direct sum of uniserial modules.
- (II) Given any two uniserial submodules U and V of a homomorphic image of M , for any submodule W of U , any non-zero homomorphism $f : W \rightarrow V$ can be extended to a homomorphism $g : U \rightarrow V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

A module M_R satisfying only condition (I) is called a *QTAG*-module. The study of *QTAG*-modules was initiated by Singh [11]. This is a very fascinating structure that has been the subject of research of many authors. Different notions and structures of *QTAG*-modules have been studied, and a theory was developed, introducing several notions, interesting properties, and different characterizations of submodules. Many interesting results have been obtained, but there is still a lot to explore.

Everywhere in the text of the present article; let it be agreed that all the rings are associative with unity ($1 \neq 0$) and modules are unital *QTAG*-modules. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup \left\{ d \left(\frac{yR}{xR} \right) \mid y \in M, x \in yR \text{ and } y \text{ uniform} \right\}$ are the exponent and height of x in M , respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k . Let us denote by M^1 , the submodule of M , containing elements of infinite height. As defined in [5], the module M is *h*-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module M is *h*-reduced if it does not contain any *h*-divisible submodule. In other words, it is free from the elements of infinite height. The module M is said to be bounded, if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$.

A submodule N of M is *h*-pure [3] in M if $N \cap H_n(M) = H_n(N)$, for every integer $n \geq 0$. A submodule $B \subseteq M$ is a basic submodule [5] of M , if B is *h*-pure in M , $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is *h*-divisible. A submodule $N \subseteq M$ is said to be high [4], if it is a complement of M^1 i.e. $M = N \oplus M^1$. The sum of all simple submodules of M is called the socle of M and is denoted by $Soc(M)$. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α , $f_M(\alpha)$ is the α^{th} -*Ulm* invariant of M (see [6]) and it is equal to $g(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M)))$.

Imitating [8], the submodules $H_k(M)$, $k \geq 0$ form a neighborhood system of zero, thus a topology known as h -topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h -topology if $\overline{N} = N$.

It is interesting to note that almost all the results which hold for TAG -modules are also valid for $QTAG$ -modules [9]. Many results of this paper are the generalization of [10]. Our notations and terminology generally agree with those in [1] and [2].

2. ELEMENTARY RESULTS

We start here with a recollection of the following notions from [7].

Definition 2.1. A basic submodule B^u of a $QTAG$ -module M is said to be an upper basic submodule if $g(M/B^u) = \min\{g(M/B) \mid B \text{ is a basic submodule of } M\}$.

Definition 2.2. A basic submodule B^l is said to be a lower basic submodule of M if $g(M/B^l) = \text{fin } g(M) = \min\{H_k(M)\}$.

To develop the study, we need to prove some elementary but helpful lemmas.

Lemma 2.1. If M is a $QTAG$ -module without elements of infinite height such that every basic submodule of M is both an upper and lower basic submodule of M , and such that $\text{fin } g(M) = g(M)$. Then M cannot be decomposed as $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules, and $g(M_1) < g(M)$.

Proof. Suppose such a decomposition of M does exist, and let B be a basic submodule of M_1 . Now $B \oplus M_2$ is a basic submodule of M and

$$g(M/(B \oplus M_2)) = g(M_1/B) + g(M_2/M_2) = g(M_1/B) \leq g(M_1) < g(M).$$

Since $\text{fin } g(M) = g(M)$ there exists a basic submodule B' of M such that $g(M/B') = g(M)$. But these two facts contradict the hypothesis that every basic submodule of M is both an upper and lower basic submodule of M .

Lemma 2.2. Let M be a $QTAG$ -module without elements of infinite height. Suppose $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules, and suppose that every basic submodule of M_1 is both an upper and lower basic submodule of M_1 . If $\text{fin } g(M_1) = g(M_1)$,

$g(M_2) < g(M_1)$, and B is a basic submodule of M_1 , then $B \oplus M_2$ is an upper basic submodule of M .

Proof. Suppose that $B \oplus M_2$ is not an upper basic submodule of M , and let B^u be an upper basic submodule of M . Let $M = S \oplus P_1$ and $B^u = P_1 \oplus P_2$ where $g(S) = \max\{\aleph_0, g(M/B^u)\}$. If $g(S) \leq \aleph_0$ then we get that M is a direct sum of uniserial modules, and therefore M_1 is a direct sum of uniserial modules. Thus M_1 must be bounded since each of its basic submodules is both an upper and lower basic submodule. Therefore $B = M_1$ and so $B \oplus M_2 = M_1 \oplus M_2$ is an upper basic submodule of M . Now we assume that $\aleph_0 < g(S) = g(M/B^u) < g(M/(B \oplus M_2)) \leq g(M_1)$. Now we write $M = S \oplus Q_1 \oplus Q_2$ where $P_1 = Q_1 \oplus Q_2$, and $S \oplus Q_1$ contains M_2 and $g(S \oplus Q_1) < g(M_1)$. But we know that $M_1 \cong M/M_2 \cong [(S \oplus Q_1)/M_2] \oplus Q_2$ which contradicts Lemma 2.1 when applied to M_1 , therefore $B \oplus M_2$ must be an upper basic submodule of M .

Lemma 2.3. *Let M be a QTAG-module without elements of infinite height such that $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules, fin $g(M_1) = g(M_2)$, and every basic submodule of M_1 is both an upper and lower basic submodule of M_1 . If B is a basic submodule of M_1 , then $B \oplus M_2$ is an upper basic submodule of M .*

Proof. Suppose that $B \oplus M_2$ is not an upper basic submodule of M , and let B^u be an upper basic submodule of M . As in Lemma 2.2 we can assume that $g(M/B^u) > \aleph_0$. Write $M = S \oplus P_1$ and $B^u = P_1 \oplus P_2$ where $g(S) = g(M/B^u) < g(M/(B \oplus M_2)) \leq g(M_1)$. Now we can write $M = M_1 \oplus R_1 \oplus R_2$ where $M_2 = R_1 \oplus R_2$ and $M_1 \oplus R_1$ contains S and $g(R_1 + S) < g(M_2)$. Consider the module $M_1 \oplus R_1$. By Lemma 2.2 we know that $B \oplus R_1$ is an upper basic submodule of $M_1 \oplus R_1$. But $M_1 \oplus R_1$ contains S which is a summand of M so that we can write $M_1 \oplus R_1 = S \oplus [(M_1 \oplus R_1) \cap P_1]$. Let $T = (M_1 \oplus R_1) \cap P_1$. Now observe that $g((M_1 \oplus R_1)/(P_2 \oplus T)) = g(S/P_2) \leq g(S) < g(M_1)$. Since fin $g(M_1) = g(M_1)$, and every basic submodule of M_1 is both an upper and lower basic submodule of M_1 with $g(M_1/B) = g(M_1)$ which contradicts $B \oplus R_1$ being an upper basic submodule of $M_1 \oplus R_1$. Thus $B \oplus M_2$ must be an upper basic submodule of M .

Lemma 2.4. *Let M be an h -reduced QTAG-module such that every basic submodule of M is an upper and lower basic submodule of M . Suppose $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules. Then M_1 has the property that each of its basic submodules is both an upper and lower basic submodule of M_1 .*

Proof. Suppose there exists two basic submodules B_1 and B_2 of M_1 such that $g(M_1/B_1) \neq g(M_1/B_2)$. Then $B_1 \oplus M_2$ and $B_2 \oplus M_2$ are basic submodules of M such that $g(M/(B_1 \oplus M_2)) = g(M_1/B_1) \neq g(M_1/B_2) = g(M/(B_2 \oplus M_2))$, and this contradicts the hypothesis on M .

Lemma 2.5. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where $\text{fin } g(M) = g(M)$, $\text{fin } g(M_1) = g(M_1)$, $\text{fin } g(M_2) = g(M_2)$, and M_1 and M_2 have the property that every basic submodule is an upper and lower basic submodule. If $M = M_3 \oplus M_4$ where $g(M_3) < g(M)$, and M_4 is a direct sum of uniserial modules, then $g(M) = g(M_1) = g(M_2)$.*

Proof. Suppose that $g(M_1) < g(M)$, then $g(M_2) = g(M)$. Now we write $M = M_3 \oplus T_1 \oplus T_2$ where $M_3 \oplus T_1$ contains M_1 , and $g(M_3 \oplus T_1) < g(M)$. This is possible since $g(M_3) < g(M)$, and $g(M_1) < g(M)$. Now notice that $M_2 \cong M/M_1 \cong [(M_3 \oplus T_1)/M_1] \oplus T_2$, and $g((M_3 \oplus T_1)/M_1) \leq g(M_3 \oplus T_1) < g(M) = g(M_2)$. But this contradicts Lemma 2.1 when applied to M_2 , therefore we have $g(M) = g(M_1) = g(M_2)$.

Lemma 2.6. *Let M be a QTAG-module without elements of infinite height and suppose that $M = M_1 \oplus M_2$, where $\text{fin } g(M_1) = g(M_1)$, and $\text{fin } g(M_2) = g(M_2)$, $\text{fin } g(M) = g(M)$, and every basic submodule of M_1 or M_2 is both an upper and lower basic submodule. Let B_1 and B_2 be basic submodules of M_1 and M_2 respectively. If either $g(M_1) < g(M)$ or $g(M_2) < g(M)$, then $B_1 \oplus B_2$ is an upper basic submodule of M .*

Proof. Assume that $B_1 \oplus B_2$ is not an upper basic submodule of M . Let B^u be an upper basic submodule of M . Write $M = M_3 \oplus M_4$ where M_4 is a direct sum of uniserial modules, and $g(M_3) = \max(\aleph_0, g(M/B^u))$. If $g(M_3) \leq \aleph_0$, then M is a direct sum of uniserial modules. But this means that M_1 and M_2 are bounded since the bounded direct sum of uniserial modules are the only direct sums of uniserial modules which have the property that every basic submodule is both an upper and lower basic submodule. Thus M is a bounded module and hence only basic submodule which contradicts the assumption that $B_1 \oplus B_2$ is not an upper basic submodule of M . Therefore $\aleph_0 < g(M/B^u)$, and $g(M_3) = g(M/B^u)$.

Since $B_1 \oplus B_2$ is not an upper basic submodule of M , we get that $g(M/B^u) < g(M)$, but this contradicts Lemma 2.5. Thus $B_1 \oplus B_2$ must be an upper basic submodule of M .

3. MAIN RESULTS

Through the proceeding series of lemmas, we are in a position to proceed by proving the following.

Theorem 3.1. *Let M be a QTAG-module without elements of infinite height such that $M = M_1 \oplus M_2$, where M_2 is a direct sum of uniserial modules. If B^u is an upper basic submodule of M_1 , then $B^u \oplus M_2$ is an upper basic submodule of M .*

Proof. If M_1 is a finitely generated, module then $B^u \oplus M_2 = M$ since B^u is basic submodule in M_1 , thus $B^u \oplus M_2$ is upper basic submodule. Since B^u is an upper basic submodule of M_1 , we can write, $M_1 = S_1 \oplus T_1$ and $B^u = T_1 \oplus T_2$ where $g(S_1) = \max(\aleph_0, g(M_1/B^u))$, and every basic submodule of S_1 is both an upper and lower basic submodule of S_1 . As in Lemma 2.2, we assume that $g(M_1/B^u) > \aleph_0$. Now if $\text{fin } g(S_1) < g(S_1)$ then we write $S_1 = S_2 \oplus S_3$ and $T_1 = K \oplus S_3$ where $\text{fin } g(S_2) < g(S_2)$. By Lemma 2.4 every basic submodule of S_2 is both an upper and lower basic submodule of S_2 . Thus $M = S_2 \oplus S_3 \oplus T_2 \oplus M_2$ where $S_3 \oplus T_2 \oplus M_2$ is a direct sum of uniserial modules, and hence, by Lemma 2.3 we have that $K \oplus S_3 \oplus T_2 \oplus M_2$ is an upper basic submodule of M , but $K \oplus S_3 \oplus T_2 \oplus M_2 = B^u \oplus M_2$.

With the last statement in hand, we establish the following corollaries about decomposition of QTAG-modules.

Corollary 3.1. *Let M be a QTAG-module without elements of infinite height such that $M = M_1 \oplus M_2 = M_3 \oplus M_4$, where $\text{fin } g(M_1) = g(M_1)$ and $\text{fin } g(M_3) = g(M_3)$. Suppose that M_2 and M_4 are direct sums of uniserial modules, and every basic submodule of M_1 and M_3 is an upper and lower basic submodule. Then $\text{fin } g(M_1) = \text{fin } g(M_3)$.*

Proof. Let B_1 and B_2 be basic submodule of M_1 and M_3 respectively. Now we have $\text{fin } g(M_1) = g(M_1/B_1) = g(M/(B_1 \oplus M_2)) = g(M/(B_2 \oplus M_4)) = \text{fin } g(M_3)$, since $B_1 \oplus M_2$ and $B_2 \oplus M_4$ are upper basic submodules of M by Theorem 3.1.

Corollary 3.2. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where $\text{fin } g(M_1) = g(M_1)$ and $g(M_2) < g(M_1)$, and every basic submodule of M_1 is both an upper and lower basic submodule of M_1 . If B^u is an upper basic submodule of M_2 , and B is a basic submodule of M_1 , then $B \oplus B^u$ is an upper basic submodule of M .*

Proof. If M is finitely generated, the proof is trivial. Now we write $M_2 = S \oplus T_1$ and $B^u = T_1 \oplus T_2$ where every basic submodule of S is both an upper and lower basic submodule

of S . We can also assume that $\text{fin } g(S) = g(S)$. Consider the module $M_1 \oplus S$ satisfies the hypothesis of Lemma 2.6, and so $B \oplus T_2$ is an upper basic submodule of $M_1 \oplus S$. Now by Theorem 3.1, we have $B \oplus T_1 \oplus T_2 = B \oplus B^u$ is an upper basic submodule of M .

And so, we are ready to prove the following.

Proposition 3.1. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where $\text{fin } g(M_1) = g(M_1)$ and $\text{fin } g(M_2) = g(M_2)$, $\text{fin } g(M) = g(M)$, and every basic submodule of M_1 and M_2 is both an upper and lower basic submodule. If B_1 and B_2 are basic submodules of M_1 and M_2 respectively, then $B_1 \oplus B_2$ is an upper basic submodule of M .*

Proof. By Lemma 2.6 we assume that $g(M_1) = g(M_2) = g(M)$. Suppose that $B_1 \oplus B_2$ is not upper basic submodule of M , and let B^u be an upper basic submodule of M . Now we write $M = S \oplus Q_1$ and $B^u = Q_1 \oplus Q_2$ where S has the property that every basic submodule of S is both an upper and a lower basic submodule of S , and $g(S) = \max(\aleph_0, g(M/B^u))$. As in the proof of Lemma 2.6 we can assume that $\aleph_0 < g(S) = g(M/B^u) < g(M)$. We may also assume that $\text{fin } g(S) = g(S)$. Consider the module $M_1 + S$. Since $M_1 + S$ contains the modules M_1 and S , both of which are summands of M , we have $M_1 + S = M_1 \oplus [(M_1 + S) \cap T]$, and $M_1 + S = S \oplus [(M_1 + S) \cap Q_1]$. Let $U = (M_1 + S) \cap T$, and let B_1^u be an upper basic submodule of U . Observing that

$$\begin{aligned} g(S) &= g(S/Q_2), \\ &= g((S + M_1)/(Q_2 \oplus [(M_1 + S) \cap Q_1])), \\ &= g((M_1 + S)/(B_1 \oplus B_1^u)), \\ &= g(M_1/B_1) + g(U/B_1^u) \end{aligned}$$

Now $g(S) = g(M_1/B_1) + g(U/B_1^u) = g(M_1) = g(U/B_1^u)$, since $\text{fin } g(M_1) = g(M_1)$, and every basic submodule of M_1 is both an upper and a lower basic submodule of M_1 . Thus we have that $g(M_1) \leq g(S)$, but this a contradiction since $g(S) < g(M) = g(M_1)$. Therefore $B_1 \oplus B_2$ must be an upper basic submodule of M .

Theorem 3.2. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, and let B_1^u and B_2^u be upper basic submodules of M_1 and M_2 respectively. Then $B_1^u \oplus B_2^u$ is an upper basic submodule of M .*

Proof. If either M_1 or M_2 is finitely generated, then by Theorem 3.1, $B_1^u \oplus B_2^u$ is an upper basic submodule in M . Now we write $M_1 = S_1 \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where every basic submodule of S_1 is both an upper and a lower basic submodule of S_1 . Similarly we can write $U = V_1 \oplus T_1$ and $B_2^u = T_1 \oplus T_2$ where every basic submodule of V_1 is both an upper and a lower submodule of V_1 . Now we have $S_1 = S_2 \oplus P_3$ and $P_2 = P_3 \oplus W_1$ where $\text{fin } g(S_2) = g(S_2)$. Similarly we have $V_1 = V_2 \oplus T_3$ and $T_2 = T_3 \oplus W_2$ where $\text{fin } g(V_2) = g(V_2)$. Therefore by Lemma 2.4 we get that S_2 and V_2 have the property that every basic submodule is both an upper and a lower basic submodule. Thus $M = (S_2 \oplus V_2) \oplus (P_1 \oplus P_3 \oplus T_1 \oplus T_3)$ and by applying Theorem 3.1 and Proposition 3.1 the proof is completed.

For freely use in the sequel, we state the following.

Corollary 3.3. *Let M be a QTAG-module without elements of infinite height. Suppose that $M = M_1 \oplus M_2$, where every basic submodule of M_1 or M_2 is both an upper and a lower basic submodule, and suppose that $\text{fin } g(M) = g(M)$. Then every basic submodule of M is an upper and lower basic submodule of M .*

Proof. Let B_1^u and B_2^u be upper basic submodules of M_1 and M_2 respectively. By Theorem 3.2, we have $B_1^u \oplus B_2^u$ is an upper basic submodule of M . Notice that $\text{fin } g(M) = \text{fin } g(M_1) + \text{fin } g(M_2) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M/(B_1^u \oplus B_2^u))$, and we are done.

Now we are able to prove the following.

Theorem 3.3. *Let M be an h -reduced QTAG-module, and let N be a high submodule of M . If B^u is an upper basic submodule of N , then B^u is an upper basic submodule of M .*

Proof. Suppose that B^u is not upper basic submodule in M , and let B_1^u be an upper basic submodule of M and K a high submodule of M containing B_1^u with $g(M/B_1^u) < g(M)$. We have two cases to consider.

Case (i). Suppose that $g(K/B_1^u) \leq \aleph_0$, then M is a Σ -module and $g(M/N) = g(M/K)$. Since B^u is an upper basic submodule of N and M is a Σ -module, then $B^u = N$, and thus B^u is an upper basic submodule of M .

Case (ii). Suppose that $g(K/B_1^u) > \aleph_0$, then $g(M/B^u) > \aleph_0$ and we have $M = S \oplus P_1$ and $B_1^u = P_1 \oplus P_2$ where $g(S) = g(M/B_1^u)$. Now K contains P_1 and hence $K = P_1 \oplus T$ where $T = S \cap K$. Let $\phi : M \rightarrow M/M^1$ be the natural quotient map such that $N \cong \phi(N)$, $K \cong \phi(K)$, $B_1^u \cong \phi(B_1^u)$, and $\text{Soc}(\phi(N)) = \text{Soc}(\phi(K))$. Since $K = P_1 \oplus T$ we have $\phi(K) = \phi(B_1^u) \oplus \phi(T)$. Now $\text{Soc}(\phi(B_1^u)) = \cup_{i=1}^{\infty} Q_i$ where Q_i is a submodule of elements of bounded height in $\phi(M)$.

and consequently in $\phi(N)$ and $\phi(K)$ since both are h -pure in $\phi(M)$. Therefore there exists a basic submodule B_1 of $\phi(N)$ such that $B_1 \supset \text{Soc}(\phi(B_1^u))$. Let $B_2 = \phi^{-1}(B_1) \cap N$, since ϕ is an isomorphism between N and $\phi(N)$ and B_2 is a basic submodule of N . If $g(N/B_2) \leq \aleph_0$ an argument as in Case (i) would complete the proof. Thus assume that $g(N/B_2) > \aleph_0$ and consider

$$\begin{aligned} g(N/B_2) &= g(\phi(N)/B_1) \\ &= g(\text{Soc}(\phi(N))/B_1) \\ &= g(\text{Soc}(\phi(N))/\text{Soc}(B_1)) \\ &= g(\text{Soc}(\phi(K))/\text{Soc}(B_1)) \\ &= g(\text{Soc}(\phi(P_1)) \oplus \text{Soc}(\phi(T))/\text{Soc}(B_1)), \end{aligned}$$

but $\text{Soc}(B_1)$ contains $\text{Soc}(\phi(P_1))$. Hence $g(N/B_2) \leq g(\text{Soc}(T)) \leq g(\text{Soc}(S)) = g(M/B_1^u)$.

Notice that $M/B_2 \cong N/B_2 \oplus M/N$, and hence

$$g(M/B_2) = g(N/B_2) + g(M/N) \leq g(M/B_1^u) + g(M/N) = g(M/B_1^u) + g(M/K),$$

and since $g(M/K) \leq g(M/B_1^u)$, we have $g(M/B_2) \leq g(M/B_1^u) + g(M/B_1^u) = g(M/B_1^u)$. Therefore B_2 is an upper basic submodule of M . We assume that B^u is not upper basic submodule of M , and so $g(M/B^u) > g(M/B_2)$. Notice that $g(M/B^u) = g(M/N) + g(N/B^u)$, and $g(M/B_2) = g(M/N) + g(N/B_2)$, so that $g(N/B^u) > g(N/B_2)$ which contradicts B^u being an upper basic submodule of N . Therefore B^u is an upper basic submodule of M .

Corollary 3.4. *Let M be a QTAG-module, and let N_1 and N_2 be high submodules of M , and let B_1^u and B_2^u be upper basic submodules of N_1 and N_2 respectively. Then $g(N_1/B_1^u) = g(N_2/B_2^u)$.*

Proof. Follows easily from the proof of the last theorem.

4. SOME EXTENDED RESULTS

The purpose of the present section is to extending the results of Theorems 3.1 and 3.2. Several such structural consequences are now presented. In this view we first prove the following.

Theorem 4.1. *Let M be an h -reduced QTAG-module such that $M = M_1 \oplus M_2$. Let B_1^u and B_2^u be upper basic submodules of M_1 and M_2 respectively. Then $B_1^u \oplus B_2^u$ is an upper basic submodule of M .*

Proof. Let N_1 and N_2 be high submodules of M_1 and M_2 respectively, which contain B_1^u and B_2^u respectively. Now suppose that B_3^u and B_4^u are upper basic submodules of N_1 and N_2 respectively. By Theorem 3.2 we know that $B_3^u \oplus B_4^u$ is an upper basic submodules of $N_1 \oplus N_2$, and hence by Theorem 3.3, $B_3^u \oplus B_4^u$ is an upper basic submodule of M . Now $g(M/(B_3^u \oplus B_4^u)) = g(M_1/B_3^u) + g(M_2/B_4^u)$, and since B_3^u and B_4^u are upper basic submodules of N_1 and N_2 respectively, we have by Theorem 3.3, they are basic submodules of M_1 and M_2 respectively. Thus we know that $g(M_1/B_3^u) = g(M_1/B_1^u)$, and $g(M_2/B_4^u) = g(M_2/B_2^u)$. Therefore $g(M/(B_3^u \oplus B_4^u)) = g(M_1/B_3^u) + g(M_2/B_4^u) = g(M_1/B_1^u) + g(M_2/B_2^u) = g(M/(B_1^u \oplus B_2^u))$, and hence $B_1^u \oplus B_2^u$ is an upper basic submodule of M .

Theorem 4.2. *Let M be an h -reduced QTAG-module such that $M = M_1 \oplus M_2$ where M_2 is a direct sum of uniserial modules. Let B^u be an upper basic submodules of M_1 . Then $B^u \oplus M_2$ is an upper basic submodule of M .*

Proof. The proof follows easily from Theorem 4.1.

Lemma 4.1. *Let M be a QTAG-module such that $M = M_1 \oplus M_2$ be a direct sum of uniserial modules, and suppose that $g(M_2) < g(M)$ and that $\aleph_0 < g(M_2)$ is not a limit cardinal. Let N be an h -pure submodule of M_2 , and let $B_1 \oplus N$ be a basic submodule of M such that $g(M/(B_1 \oplus N)) > g(M_2)$. Then there exists a basic submodule $B_2 \oplus N$ such that $B_2 \oplus N$ contains $B_1 \oplus N$, and $g(M/(B_2 \oplus N)) \leq g(M_2)$.*

Proof. Consider $M/B_1 = M_3/B_1 \oplus M_4/B_1$ where M_4/B_1 is an h -reduced and contain $(B_1 \oplus N)/B_1$, and where M_3/B_1 is h -divisible. Notice that $M_3 \cap N = 0$ since $M_3 \cap B_1 = B_1$ and $B_1 \cap N = 0$. Thus $M_3 + N = M_3 \oplus N$, and as a submodule of a direct sum of uniserial modules is itself a direct sum of uniserial modules. To show that $M_3 \oplus N$ is a basic submodule, we need only prove that $M_3 \oplus N$ is h -pure, but $(M_3 \oplus N)/(B_1 \oplus N) \cong M_3/B_1$ which is h -divisible and hence $M_3 \oplus N$ is h -pure. Therefore $M_3 \oplus N$ is a basic submodule of M which contains $B_1 \oplus N$, and notice that $g(M/(M_3 \oplus N)) \leq g(M/M_3) = g(M_4/B_1)$, and $g(M_4/B_1) \leq g(M_2)$ since $g(M_2)$ is not a limit ordinal. This completes the proof.

Theorem 4.3. *Let M be a QTAG-module without elements of infinite height, and let B be a basic submodule of M . Let B^u be an upper basic submodules of M , and suppose that*

$g(M/B^u)$ is not a limit cardinal larger than \aleph_0 . Then B is contained in an upper basic submodule of M .

Proof. If $g(M/B^u) \leq \aleph_0$, then M is a direct sum of uniserial modules and hence B is contained in an upper basic submodule of M , namely, M itself. Thus we may assume that $\aleph_0 < g(S) = g(M/B^u)$, $M = S \oplus P_1$ where $B^u = P_1 \oplus P_2$ and $g(S) = g(M/B^u)$. Let S be the homomorphic image of the free module T with h -pure kernel K and we can assume $g(T) = g(S)$. Now $(P_1 \oplus T)/K \cong S \oplus P_1$, and suppose $(Q_1 \oplus K)/K \cong B$. If $g(M/B) = g(M/B^u)$ we know that B is already an upper basic submodule of M and we are done, so that we can assume that $g(M/B) > g(M/B^u) = g(S)$. Thus $g[(T \oplus P_1)/(Q_1 \oplus K)] > g(S) = g(T)$, and by Lemma 4.1 there exists a basic submodule $Q_2 \oplus K$ containing $Q_1 \oplus K$ and such that $g[(T \oplus P_1)/(Q_2 \oplus K)] = g(T)$. Let $R \cong (Q_2 \oplus K)/K$. We know that R is a basic submodule of M which contains B and R is upper basic submodule of M since $g(M/R) = g[(T \oplus P_1)/(Q_2 \oplus K)] = g(S) = g(M/B^u)$.

Theorem 4.4. *Let M be an h -reduced QTAG-module and let B be a basic submodule of M . If there exists a high submodule N of M which contains B , and an upper basic submodule B^u of N containing B , then B is contained in an upper basic submodule of M .*

Proof. If B^u is an upper basic submodule of N , then B^u is an upper basic submodule of M by Theorem 3.3.

Let \mathfrak{B} be the class of QTAG-modules which have the property that every basic submodule is contained in an upper basic submodule.

Theorem 4.5. *The class \mathfrak{B} contains all QTAG-modules which are direct sum of an h -divisible and a bounded module.*

Proof. This follows immediately from the fact that such modules have only basic submodule, and it is by necessity an upper basic submodule.

Recall that a module M is a Σ -module (see [4]) if some its high submodule is a direct sum of uniserial modules.

Theorem 4.6. *The class \mathfrak{B} contains all Σ -modules.*

Proof. Let M be Σ -module, and B be a basic submodule of M . Now B can be embedded in a high submodule of N of M , and since M is a Σ -module we get that N is an upper basic submodule of M .

Theorem 4.7. *Let M be a QTAG-module without elements of infinite height. Suppose that $\text{fin } g(M)$ is equal to its cardinality, and that B is a basic submodule of M . If $g(M/B)$ is equal to the cardinality of \overline{M} , the closure of M , then $M \in \mathfrak{B}$.*

Proof. This follows from Theorem 4.3.

Corollary 4.1. *The class \mathfrak{B} contains all closed modules.*

5. OPEN PROBLEMS

In closing, we pose the following questions of interest:

Problem 5.1. *If M is an h -reduced QTAG-module such that $M = \sum_{\alpha \in I} M_{\alpha}$, and B_{α} is an upper basic submodule of M_{α} , then is it true that $\sum_{\alpha \in I} B_{\alpha}$ is upper basic submodule of M ?*

Problem 5.2. *Does the class \mathfrak{B} defined above indeed contains all h -reduced QTAG-modules?*

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